On the density of iterated line segment intersections*

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Abstract

Given S_1, a finite set of points in the plane, we define a sequence of point sets S_i as follows: With S_i already determined, let L_i be the set of all the line segments connecting pairs of points of $\bigcup_{j=1}^{i} S_j$, and let S_{i+1} be the set of intersection points of those line segments in L_i, which cross but do not overlap. We show that with the exception of some starting configurations the set of all crossing points $\bigcup_{i=1}^{\infty} S_i$ is dense in a particular subset of the plane with nonempty interior. This region can be described by a simple definition.

1 Introduction

Given $S = S_1$, a finite set of points in the Euclidean plane, let L_1 denote the set of line segments connecting pairs of points from S_1. Next, let S_2 be the set of all the intersection points of those line segments in L_1 which do not overlap. We continue to define sets of line segments L_i and point sets S_i inductively by

$$L_i := \left\{ pq \;\middle|\; p, q \in \bigcup_{j=1}^{i} S_j \;\wedge\; p \neq q \right\},$$
$$S_{i+1} := \{ x \mid \{x\} = l \cap l' \text{ where } l, l' \in L_i \}.$$

Finally, let $S_\infty := \bigcup_{i=1}^{\infty} S_i$ denote the limit set.

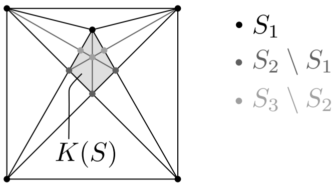

Figure 1: The first iterations S_1, S_2 and S_3 of line segment intersections and the candidate $K(S)$.

In this article we show in which region $K(S)$ the crossing points S_∞ are dense, and in which exceptional cases the crossing points are not dense in any set with non-empty interior.

Several results concerning the density of similar iterated constructions are known, see Bezdek and Pach [2], Kazdan [10], Bárány, Frankl, and Maehara [1]. Ismailescu and Radoičić [8] examined a question very similar to ours. The only difference is that they considered lines instead of line segments. They proved by applying nice elementary methods that with the exception of two cases the crossing points are dense in the whole plane. Hillar and Rhea [7] independently proved the same statement with different methods.

Our setting of line segment intersections turns out to be more difficult. It has more exceptional cases, where the crossing points are not dense in any set with non-empty interior. And in non-exceptional cases the crossing points are not dense in the whole plane but only in a particular convex region $K(S)$. In fact, if we do not have an exceptional configuration, the density of the line intersections in the whole plane is an easy consequence of the result presented here. This can be shown by the arguments displayed in Figure 4 of [8].

Our work is also motivated by another interesting problem introduced recently by Ebbers-Baumann et al. [4], namely how to embed a given finite point set into a graph of small dilation. For a given geometric graph G in the plane and for any two vertices p and q we define their *vertex-to-vertex dilation* as $\delta_G(p,q) := |\pi(p,q)|/|pq|$, where $\pi(p,q)$ is a shortest path from p to q in G and $|\cdot|$ denotes the Euclidean length. The *dilation of* G, $\delta(G)$, is the maximum dilation of any two vertices.

A geometric graph G of smallest possible dilation $\delta(G) = 1$ is called *dilation-free.* We will give a list of all cases of dilation-free graphs in the plane in Section 2. Given a point set S in the plane, the *dilation of* S is defined by $\Delta(S) := \inf\{\, \delta(G) \mid G = (V,E) \text{ planar graph}, S \subset V \}$. Determining $\Delta(S)$ seems to be very difficult. The answer is even unknown if S is a set of five points placed evenly on a circle. However, Ebbers-Baumann et al. [4] were able to prove $\Delta(S) \leq 1.1247$ for every finite point set $S \subset \mathbb{R}^2$ and they showed lower bounds for some special cases.

A natural idea for embedding S in a graph of small dilation is to try to find a geometric graph $G = (V,E)$, $S \subseteq V$, such that $\delta_G(p,q) = 1$ for every $p, q \in V$. Now, suppose we have found such G. Obviously, for every pair $p, q \in S$, the line segment pq must be a part of G. Since G must be planar, every intersection point of these line segments must also be in V and so on.

*The full version of this article is available as technical report [5] via internet. Parts of the results stem from Sanaz Kamali Sarvestani's master's thesis [9].

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If this iteration produces only finitely many intersection points, i.e. the set S_∞ defined at the very beginning of this article is finite, we have a planar graph $G = (V, E)$ with $S \subset V$, $V \setminus S$ finite and $\delta(G) = 1$ thus $\Delta(S) = 1$. This shows that $|S_\infty| < \infty$ can only hold if S is a subset of the vertices of a dilation-free graph. We call those point sets *exceptional configurations.* Note that $\Delta(S) = 1$ could still hold for other sets. There could be a sequence of proper geometric graphs whose dilation does not equal 1 but converges to 1.

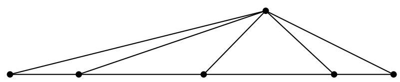
Our main result, Theorem 8, shows that in all the cases where S is not an exceptional configuration, S_∞ is dense in a region with non-empty interior. Very recently Klein and Kutz [11] proved a special case of this theorem and used it to prove the first non-trivial lower bound $\Delta(S) \geq 1.0000047$ which holds for every non-exceptional finite point set S.

2 Exceptional Configurations and the Candidate

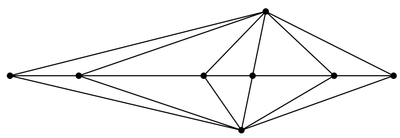
Here, we list all cases of dilation-free graphs. They can also be found at Eppstein's Geometry Junkyard [6]. It can be proven by case analysis that these are all possibilities. The *exceptional configurations* are the subsets of the vertices of such graphs.

(*i*) n points on a line

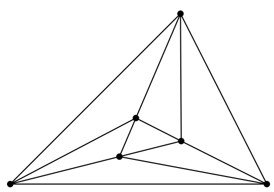
(*ii*) $n - 1$ points on a line, one point not on this line



(*iii*) $n - 2$ points on a line, two points on opposite sides of this line[1]



(*iv*) a triangle (i.e. three points) nested in the interior of another triangle. Every pair of two inner points is collinear with one outer point.



Next, we define the region $K(S)$, cf. Figure 1. Until we prove that S_∞ is dense in $K(S)$, we call $K(S)$ the candidate.

Definition 1 *The* candidate $K(S)$ *is defined as the intersection of those closed half planes which contain all the starting points except for at most one point:*

$$K(S) := \bigcap_{p \in S} \bigcap_{H \supset S \setminus \{p\}} H$$

The second intersection is taken over all closed half planes H which contain $S \setminus \{p\}$.

It is not difficult to prove that the candidate is a convex polygon whose vertices belong to $S_1 \cup S_2$, and that every intersection point lies inside of the candidate, that is $S_\infty \setminus S \subset K(S)$, see the full paper [5].

3 Density in a Triangle

Lemma 1 *(Main Lemma) Let the starting configuration $S = \{A, B, C, D, E\}$ be as follows: 3 points are the vertices of the triangle $\triangle ABC$, another 2 points D and E are on different sides of this triangle (see Figure 2), then S_∞ is dense in a triangle.*

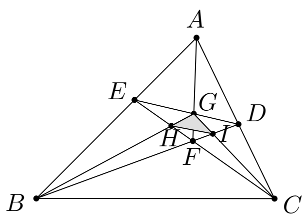


Figure 2: S_∞ is dense in $\triangle GHI$.

Proof. We use arguments from projective geometry; see Bourbaki [3] for an introduction. Consider the Euclidean plane $\mathcal{A}$ as embedded in the projective plane $\mathbb{P}^2$. The complement of the projective line through BC in $\mathbb{P}^2$ is an Euclidean plane $\mathcal{A}'$. On $\mathcal{A}'$ we have the same points as in $\mathcal{A}$ like shown in Figure 3.

We use this simple topological fact: A set A is dense in a set B with respect to a topological subspace $Y \subset X$ where $A \subset B \subset Y$ iff A is dense in B with respect to the whole space X. That means to prove, that S_∞ is dense in a triangle in the Euclidean plane $\mathcal{A}$, it is sufficient to show, that S_∞ is dense in the same triangle in the projective plane and it is also equivalent to prove, that S_∞ is dense in that triangle with respect to $\mathcal{A}'$. In this case we want ro prove

[1] Let p_1 and p_2 be the two points on opposite sides of the line, and let $p_3, \ldots, p_n$ be the other points. If the segment p_1p_2 intersects with the convex hull $\mathrm{ch}(\{p_3, \ldots, p_n\})$, the intersection point must be a vertex of the dilation-free graph. However, it does not have to be part of the corresponding exceptional configuration.

that S_∞ is dense in $\triangle GHI$ with $G := ED \cap AF$, $H := BG \cap EF$ and $I := CG \cap FD$.

In $\mathcal{A}'$ we have $AD \parallel EF$, because the lines through AD and EF in $\mathcal{A}$ cross each other in C and $AE \parallel DF$, because the lines through them cross each other in B respectively. Hence G is the midpoint of DE, I is the midpoint of DF and H is the midpoint of EF. But for this special case we can prove with elementary methods that the intersection points are dense in $\triangle GHI$ (see Section 3 of the full paper [5]). Hence S_∞ is dense in $\triangle GHI$ in $\mathcal{A}$. □

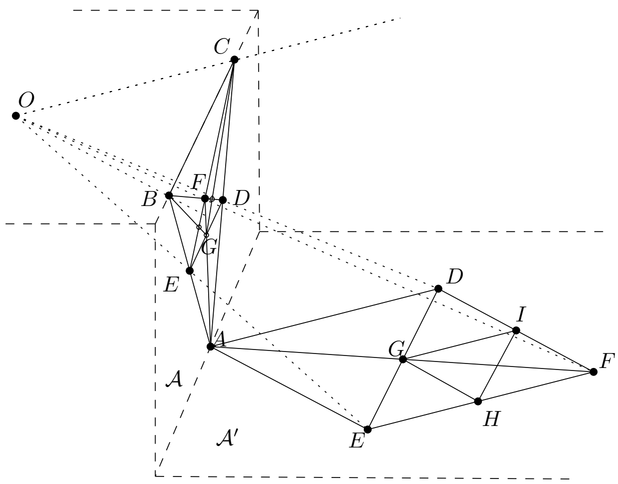


Figure 3: A useful projection of $\triangle ABC$

This implies the first partial result, proved as Corollary 7 in the full version [5]:

Corollary 2 *Let S_1 be a set of $n > 4$ points in convex position in the plane no three of them on a line. Let S_i be the sets defined as above, then $S_\infty = \bigcup_{i=1}^{\infty} S_i$ is dense in a triangle.*

We generalize this statement as follows.

Lemma 3 *For any non exceptional configuration, there exists a triangle, in which S_∞ is dense.*

This can be proved by detailed case analysis and by applying the main lemma, Lemma 1 (see proof of Lemma 8 in [5]).

4 Density in the Candidate

We mention the following two techical tools without proof (see the proofs in Section 5 of [5]).

Lemma 4 *Let L, M and N be three distinct points such that M lies on the line segment LN and let a and b be two rays, emanating from M on the same side of the line through L and N. Let K_0 be a point on the ray a (cf. Figure 4). We define a sequence of*

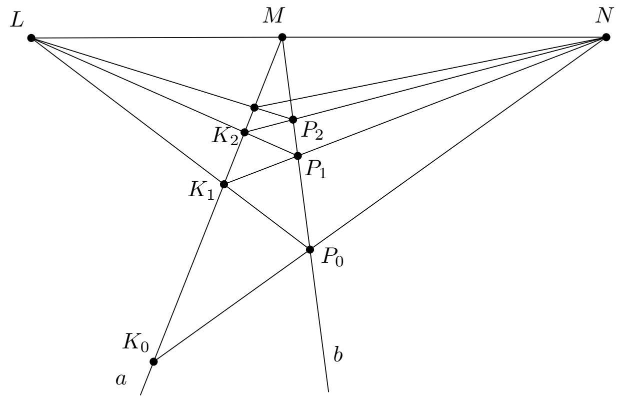


Figure 4: The point sequences $(P_i)_{i\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{N}}$ converge to M.

points as follows: $P_i := b \cap NK_i$, $K_{i+1} := LP_i \cap a$. Then the point sequence $(P_i)_{i\in\mathbb{N}}$ converges to M on b and the point sequence $(K_i)_{i\in\mathbb{N}}$ converges to M on a.

Corollary 5 *Consider Figure 5: Let L and N be two points, and this time let M be a point not on the line LN but such that the line which goes through M and is perpendicular to LN meets the line segment LN. Let H be the half plane bounded by the line which passes LM and which does not contain N and similarly let H' be the half plane, bounded by the line which passes MN and which does not contain L. Let a and b be two rays emanating from M and lying in $H \cap H'$ and not on the boundary. If K_0 is a point on the ray a and we define point sequences $(P_i)_{i\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{N}}$ analogously to Lemma 4, then these sequences converge to M.*

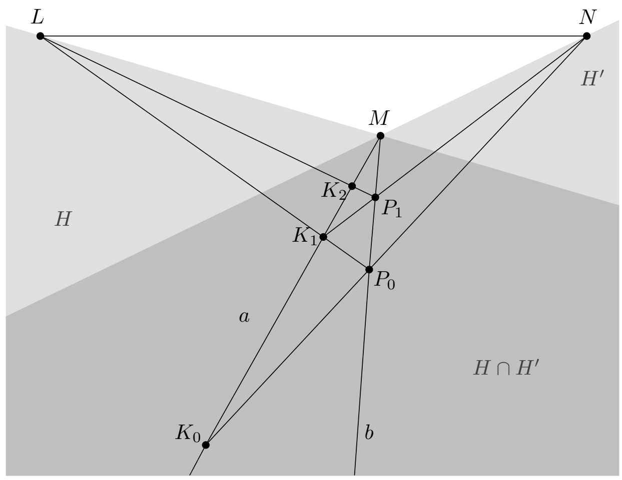


Figure 5: The point sequences $(P_i)_{i\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{N}}$ converge to M.

Definition 2 *Let $\mathcal{C}$ be a convex polygon and let P be a point outside of $\mathcal{C}$. We call the two rays emanating from P, which touch the boundary of $\mathcal{C}$,* tangents *of $\mathcal{C}$ through P. Let A and B be those two vertices of $\mathcal{C}$, which are the first to be touched by the tangents. We define the* visibility cone *of P with respect to $\mathcal{C}$, $V(P,\mathcal{C})$, as follows:*

$$V(P,\mathcal{C}) := \triangle ABP - \mathcal{C}$$

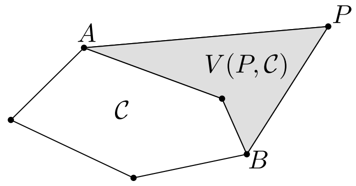


Figure 6: The visibility cone of P with respect to $\mathcal{C}$, $V(P,\mathcal{C})$

In the following we often use this simple fact (see the proof in Attachment A of [5]):

Fact 6 *A family of dense rays emanating from a fixed point generates dense intersection points on every line segment which is hit by the rays.*

Lemma 7 *Let the starting configuration S be an arbitrary finite point set. Assume that S_∞ is dense in a convex region $\mathcal{R} \subset \text{ch}(S)$ with nonempty interior. And let P be a point from S_∞ such that $P \notin \mathcal{R}$ and P is not a vertex of $\text{ch}(S)$. Then S_∞ is dense in $\text{ch}(\{P\} \cup \mathcal{R})$.*

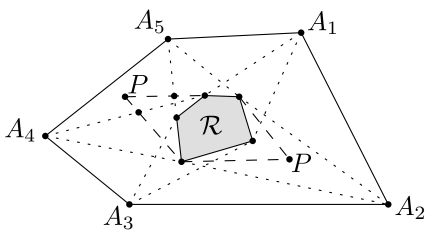


Figure 7: Two cases in the proof of Lemma 7.

Proof. Let A_i, $i = 1 \ldots n$ be the vertices of $\text{ch}(S)$.

We distinguish the following cases:

Case 1 : If the point P lies in or on the boundary of a visibility cone $V(A_i, \mathcal{R})$ for some i say i_0, then we have $V(P,\mathcal{R}) \subset V(A_{i_0}, \mathcal{R})$. Thus by Fact 6, S_∞ is dense in $V(P,\mathcal{R})$ and therefore in $\text{ch}(\{P\} \cup \mathcal{R})$.

Case 2 : Else: $V(P,\mathcal{R})$ intersects at least one $V(A_i, \mathcal{R})$.

Hence S_∞ is dense in the intersection of these visibility regions. In particular S_∞ is dense in a sub line segment of at least one tangent of $\mathcal{R}$ emanating from P. Now we can apply Corollary 5 or Lemma 4 combined with Fact 6 (P as M, the tangents as the half lines a and b, and the neighbor side of the convex hull as LM, cf. Figure 7). □

Theorem 8 *Let $S = S_1$ be a set of n points in the plane, which is not an exceptional configuration. Then S_∞ is dense in the candidate $K(S)$.*

Proof. If S is not an exceptional configuration, by Lemma 3 we know that S_∞ is dense in a triangle T. Furthermore we have seen in Section 2 that $T \subset K(S)$ and that every vertex of $K(S)$ is an intersection point. Let $P_1, P_2, \ldots, P_n$ be the vertices of $K(S)$. Then using Lemma 7 inductively yields that S_∞ is dense in $\text{ch}(\{P_1, P_2, \ldots, P_n\}) = K(S)$. □

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