On the All-Farthest-Segments Problem for a Planar Set of Points

Asish Mukhopadhyay *
e-mail: asishm@cs.uwindsor.ca

Samidh Chatterjee †
e-mail: chatte7@cs.uwindsor.ca

Benjamin Lafreniere ‡
e-mail: lafreni@uwindsor.ca

Abstract

In this note, we outline a very simple algorithm for the following problem: Given a set S of n points $p_1, p_2, p_3, \ldots, p_n$ in the plane, we have $O(n^2)$ segments implicitly defined on pairs of these n points. For each point p_i, find a segment from this set of implicitly defined segments that is farthest from p_i. The complexity of our algorithm is in $O(nh + n \log n)$, where n is the number of input points, and h is the number of vertices on the convex hull of S.

1 Introduction

Geometric optimization is a very active subarea of Computational Geometry. In this paper we study a simple geometric optimization problem. Given a set S of n points $p_1, p_2, p_3, \ldots, p_n$ in the plane, we have $O(n^2)$ segments implicitly defined on pairs of these n points. For each point p_i, find a segment from this set of implicitly defined segments that is farthest from p_i.

2 Previous work

For the *nearest* version of this problem Daescu and Luo [1], presented an $O(n \log n)$ algorithm; Duffy et al [2] presented an $O(n^2)$ algorithm for the *all-nearest* version, and also provided evidence that this might be an $O(n^2)$-hard problem. Daescu and Luo [1] also presented an $O(n \log n)$ for the *farthest* version of this problem. Here we show that the *all-farthest* version of the problem can be solved in $O(nh + n \log n))$ time, where h is the number of vertices on the convex hull of the n points.

While it is hard to provide any practical motivation for problems of this type that does not appear contrived, it is intriguing to know whether the *all-farthest* problem can be solved as efficiently as or faster than the *all-nearest* version.

*School of Computer Science, University of Windsor, Windsor,Ontario, Canada
†School of Computer Science, University of Windsor
‡School of Computer Science, University of Windsor

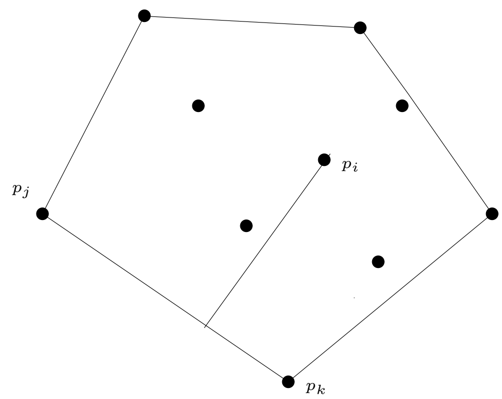

Figure 1: *Farthest distance from* p_i *to segment* (p_j, p_k) *is to an intermediate point*

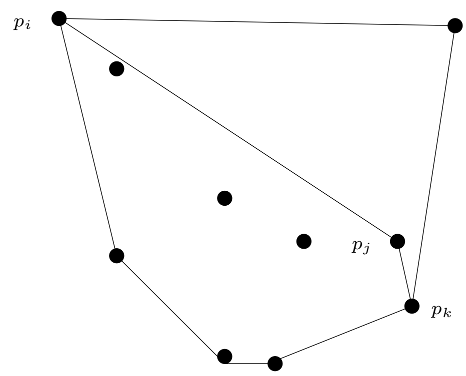

Figure 2: *Farthest distance from* p_i *to segment* (p_j, p_k) *is to an endpoint*

3 Characterization of a farthest segment

Let $\overline{p_j p_k}$ be a farthest segment of a point p_i. The farthest distance is obtained either by dropping a perpendicular from p_i to the segment $\overline{p_j p_k}$ (Fig. 1) or by joining p_i to the nearer one of the end points p_j and p_k (Fig. 2). We call these two types of farthest segments type A and type B respectively.

We design an algorithm by characterizing the two types of segments. To ensure the correctness of the arguments below, we shall assume that no three points of S are collinear.

Lemma 1 *If the segment* $\overline{p_j p_k}$ *is a type* A *farthest segment for a point* p_i *then* $\overline{p_j p_k}$ *is an edge on the convex hull of* S.

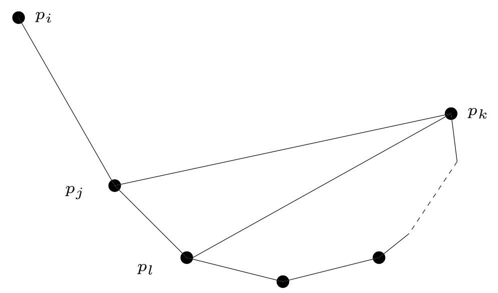

Figure 3: p_j *and* p_k *are non-adjacent convex hull vertices*

Proof: If the segment $\overline{p_j p_k}$ is not a convex hull edge, then there exists a point p_l of S in the open half-plane defined by the supporting line through p_j and p_k that does not contain p_i. This gives a segment $\overline{p_j p_l}$ that is farther from p_i than $\overline{p_j p_k}$ since $\overline{p_i p_j}$ is the hypotenuse of the right-triangle formed by p_i, p_j and the foot of the perpendicular from p_i to $\overline{p_j p_k}$. This contradicts the assumption that $\overline{p_j p_k}$ is a farthest segment of p_i. □

Lemma 2 *If the segment* $\overline{p_j p_k}$ *is a type B farthest segment for a point* p_i *then either* $\overline{p_j p_k}$ *is an edge on the convex hull of* S *or* p_j *is farthest from* p_i *among all the points that are interior to the convex hull of the point set, while* p_k *is a convex hull vertex of the given point set (Fig. 2).*

Proof: Let the farthest distance be realised by joining p_i to p_j. Our proof is in three parts, covering the mutually exclusive and exhaustive possibilities that the end points of $\overline{p_j p_k}$ are both points internal to the convex hull of S, are both convex hull vertices or one is an internal vertex while the other is a convex hull vertex.

(1) Suppose p_j and p_k are both internal to the convex hull. If this were true, consider the half-plane defined by a line through p_k orthogonal to $\overline{p_i p_k}$ that does not contain p_i. This half plane must contain a vertex p_l of the convex hull of S, giving us a segment $\overline{p_k p_l}$ that is farther from p_i than $\overline{p_j p_k}$ and a contradiction. Hence this possibility is excluded.

(2) Suppose p_j and p_k are both vertices of the convex hull. We claim that in this case $\overline{p_j p_k}$ is a convex hull edge. If otherwise, the segment $\overline{p_j p_k}$ divides the convex hull of S into two parts. Consider the convex hull boundary going from p_j to p_k that lies in the part not containing p_i (see Fig. 3). Since there is at least one convex hull vertex on this boundary, let p_l be the one closest to p_j. Then $\overline{p_l p_k}$ gives us a segment (could be of type A or Type B) that is farther from p_i than $\overline{p_j p_k}$ as the distances of all points on $\overline{p_l p_k}$ from p_i are greater than the distance from p_i to p_j. This proves our claim.

(3) p_k is a convex hull vertex and p_j is an internal vertex. We claim that in this case p_j is farthest from p_i among all internal vertices. Otherwise, let p_l be an internal point that is farther from p_i than p_j. There exists a point p_m that lies on the convex hull and is in the half-plane defined by a line through p_l orthogonal to $\overline{p_i p_l}$, not containing p_i. This gives us a segment $\overline{p_l p_m}$ farther from p_i than $\overline{p_j p_k}$ and a contradiction.

By (1), (2) and (3), we have proven that the farthest segment from p_i must either be an edge of the convex hull of S, or have one end point on the convex hull while the other end point is the farthest from p_i among all internal points. □

With these two lemmas, it is easy to design an efficient algorithm for solving this problem.

4 Algorithm

We first construct the convex hull of the point set; then the farthest-point Voronoi diagram of the interior points, if any. The complexity of these two steps is in $O(n \log n)$.

For each point p_i, we find the farthest segment as outlined in the following algorithm.

Algorithm All-farthest-segments
Input: A set of n points $p_1, p_2, \ldots, p_n$
Output: The farthest segment $\overline{p_j p_k}$ for each p_i

for each p_i **do**

Step 1: Find the fathest segment among the edges of the precomputed convex hull; record the segment and the distance.

Step 2: Locate p_i in the precomputed farthest point Voronoi diagram of the points interior to the convex hull. Let p_j be its farthest neighbor and record the distance to it from p_i. If this distance is smaller than that computed in Step 1 report the segment found in Step 1 and quit, else continue.

Step 3: Draw a line orthogonal to the segment $\overline{p_i p_j}$; the other endpoint p_k is the convex hull vertex that lies in the halfplane not containing p_i. We find this by a linear search (we can afford this!) on the convex hull boundary. Report $\overline{p_j p_k}$ as the farthest segment.

od

---

5 Analysis

The complexity of Step 1 is in $O(h)$; that of Step 2 is in $O(\log(n-h))$; while that of Step 3 is also in $O(h)$. Thus the complexity of the all farthest segment is in $O(nh + n \log n)$.

6 Future Work

It would be intereresting to extend this algorithm to finding all k-th closest segments.

7 Acknowledgement

We would like to thank the referees for their perceptive comments. The improvement and clarity of this revised version are due to their comments.

References

[1] O. Daescu and J. Luo. Proximity problems on line segments spanned by points. In *Proc. of 14th Annual Fall Workshop on Computational Geometry*, pages 9–10, 2004.

[2] H. M. K. Duffy, C. McAloney and D. Rappaport. Closest segments. In *Proc. of CCCG 2005*, pages 229–231, 2005.