Planar Point Sets with Large Minimum Convex Partitions*

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Abstract

Given a finite set S of points in the plane, a *convex partition* of S is a subdivision of the convex hull of S into nonoverlapping empty convex polygons with vertices in S. Let $G(S)$ be the minimum m such that there exists a convex partition of S with at most m faces. Let $F(n)$ be the maximum value of $G(S)$ among all the sets of n points in the plane. It is known [1] that $F(n) \geq n+2$ for $n \geq 13$. In this paper we show that, for $n \geq 4$

$$F(n) > \frac{12}{11}n - 2$$

Also, for $n \geq h \geq 3$, let $F_h(n)$ be the maximum value of $G(S)$ among all the sets of n points in which exactly h of them lie in the boundary of the convex hull. We show that

$$F_h(n) > n - \frac{15}{8}h + \sqrt{\frac{(n-h)h}{2}}$$

1 Introduction

Triangulations of point sets are one of the most studied structures in computational geometry. More general structures are easily obtained by relaxing some of the conditions that define triangulations. Let a *convex partition* of a set of points S in the plane be a decomposition of the convex hull of S into nonoverlapping convex polygons with any number of edges, such that no point of S belongs to the interior of any of the convex polygons. Equivalently, a convex partition is a tesselation of the points into empty convex polygons covering the entire convex hull of S. Intuitively, S should admit convex partitions with considerably fewer faces (polygons) than those in a triangulation, and this may be useful in applications where the complexity depends on the number of faces.

Therefore, we should try to find the minimum number $F(n)$ such that every set of n points in the plane has a convex partition with at most $F(n)$ faces.

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In the open-problem session of CCCG-1998, J. Urrutia [5] conjectured that $F(n) \leq n+1$. Later, in 2001, O. Aichholzer and H. Krasser [1] showed that $F(n) \geq n+2$ (for $n \geq 13$). The best known upper bound, $F(n) \leq \frac{10n-18}{7}$, is due to V. Neumann-Lara et al. [4]. Other related work on convex partitions include the study of simultaneous flips (see [3]) and an algorithm by T. Fevens et al. [2] that computes in polynomial time the minimum convex partition provided that the points lie on the boundaries of a fixed number of nested convex hulls.

2 Definitions and Strict Monotonicity

For a finite set S of points in the plane, let $G(S)$ be the number of faces in a convex partition of S with a minimum number of faces.

For $n \geq h \geq 3$, let $F_h(n)$ be the maximum value of $G(S)$ among all the sets S with n points of which exactly h are *extreme*, i.e., lie on the boundary of the convex hull.

Let $F(n)$, $n \geq 3$, be the maximum value of $G(S)$ when $|S| = n$, so $F(n) = \max\{F_h(n) : 3 \leq h \leq n\}$.

The following results show that the functions $F_h(n)$ and $F(n)$ are strictly increasing.

Proposition 1 *$F_h(n+k) \geq F_h(n)+k$, for $3 \leq h \leq n$, $0 \leq k$.*

Proof. It is enough to show that $F_h(n+1) > F_h(n)$.

Let S be a set of n points, h of them extreme, such that $G(S) = F_h(n)$. Let p, p' be contiguous extreme points of S, see Figure 1. Take $q \notin S$ satisfying

(i) q is in the interior of the convex hull of S.

(ii) q is an extreme point of $S-\{p\}$ and $S-\{p'\}$.

(iii) for every $r_1, r_2 \in S-\{p\}$, p and q lie on the same side of the line through r_1 and r_2.

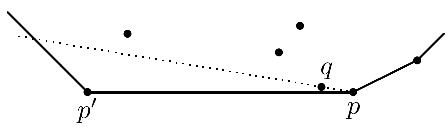

Figure 1: Points as in Proposition 1

Conditions (i) and (ii) imply that every convex partition of $S \cup \{q\}$ contains the triangle with vertices p, q, p' as one of its faces. And (iii) implies that every convex partition of $S \cup \{q\}$ can be transformed into a convex partition of S by first joining to p every vertex adjacent to q and then removing q. Since the triangle pqp' is transformed into the edge pp', the result follows by performing this transformation to a minimum convex partition of $S \cup \{q\}$. □

Corollary 2 $F(n+k) \geq F(n) + k$, *for* $n \geq 3, k \geq 0$.

3 The Arrangement

Given two points o_1, o_2 located on the y-axis, it is not difficult to show by induction on $k \geq 0$ that it is possible to arrange $k(k+1)/2$ points $\{p_{i,j}\}_{1 \leq i \leq j \leq k}$ on k vertical layers $V_j = \{p_{1,j}, p_{2,j}, \ldots, p_{j,j}\}$ and k horizontal layers $H_i = \{p_{i,i}, p_{i,i+1}, \ldots, p_{i,k}\}$, as schematically shown in Figure 2, in such a way that the following properties hold:

(1) Each horizontal layer H_i is concave upward and the y-coordinate of $p_{i,j}$ decreases as j increases.

(2) Each vertical layer V_i is concave to the left and the x-coordinate of $p_{i,j}$ increases with i.

(3) Every point in V_{j+2} lies above the line through $p_{i,j}$ and $p_{i,j+1}$, for $i \in \{1, \ldots, j\}$, $j < k-1$.

(4) Every point in V_{j+1} lies below the line through $p_{i,i}$ and $p_{i+1,j}$, for $i \in \{1, \ldots, k-2\}$, $i < j$.

(5) V_{j+1} lies below the line through o_1 and $p_{1,j}$ and above the line through o_2 and $p_{j,j}$, for $j < k$.

Additionally, let A be any nonempty set of points satisfying:

(3') A lies above the line through $p_{i,k-1}$ and $p_{i,k}$, $i < k$.

(4') A lies below the line through $p_{i,i}$ and $p_{i+1,k}$, $i < k$.

(5') A lies below the line through o_1 and $p_{1,k}$ and above the line through o_2 and $p_{k,k}$.

Let $S = \{o_1, o_2\} \cup \{p_{i,j}\}_{1 \leq i \leq j \leq k} \cup A$, and take any convex partition Π of S.

For $1 \leq i \leq j \leq k$, let $C_{i,j}$ be the convex polygon in Π located immediately below $p_{i,j}$, so $C_{i,j}$ is the unique polygon in the partition containing the point $p_{i,j} - (0, \varepsilon)$, for very small ε. Note that $C_{i,k}$ is well defined because A is not empty.

Proposition 3 $C_{i_1,j_1} \neq C_{i_2,j_2}$ *if* $(i_1, j_1) \neq (i_2, j_2)$.

Proof. By contradiction.

Suppose $C_{i_1,j_1} = C_{i_2,j_2} = C$ with $(i_1, j_1) \neq (i_2, j_2)$. Clearly, both p_{i_1,j_1} and p_{i_2,j_2} are upper extreme points of C, they lie on the upper boundary of the convex hull of C. Without loss of generality assume that p_{i_1,j_1} appears before p_{i_2,j_2} when the extreme upper points of C are listed from left to right. This assumption implies that $j_1 \leq j_2$.

Note that p_{i_1,j_1} cannot be the leftmost point among the extreme upper points of C, because then the other points of C belong to $V_{j_1+1} \cup \ldots \cup V_k \cup A$ (recall the x-coordinates of points in V_{j_1} below p_{i_1,j_1} are less than the x-coordinate of p_{i_1,j_1}), but then C cannot be the polygon immediately below p_{i_1,j_1}.

Let l be the leftmost upper extreme point of C, so we know $l \neq p_{i_1,j_1}$ and $l \in \{o_1\} \cup V_1 \cup \ldots \cup V_{j_1}$. Similarly, the rightmost point r among the extreme upper points of C cannot be p_{i_2,j_2}, so it belongs to $V_{j_2+1} \cup \ldots \cup V_k \cup A$.

By (2) we see that necessarily $j_1 < j_2$, since if $j_1 = j_2$ then l cannot lie below the line through p_{i_1,j_1} and p_{i_2,j_2}. Also, it is clear that $i_1 < i_2$ since if $i_1 \geq i_2$ then r cannot lie below the line through p_{i_1,j_1} and p_{i_2,j_2}, by (3).

If $l = o_1$ then (5) and (5') imply that p_{1,j_2} lies above the line through o_1 and r, but $i_2 > i_1 \geq 1$, so p_{1,j_2} lies in the interior of C, a contradiction.

If $l = p_{i_0,j_0}$, $j_0 \leq j_1$, then again by (3) we must have $i_0 < i_1$. Therefore, by (4), p_{i_2-1,j_2} lies above the line through p_{i_0,j_0} and r, since $i_0 < i_2 - 1$ (because $i_0 < i_1 < i_2$), so p_{i_2-1,j_2} belongs to the interior of C, a contradiction. □

A similar argument shows that the sets $C_{i,j}$ cannot contain any point in A, unless $j = k$.

Proposition 4 $C_{i,j} \cap A = \emptyset$ *if* $j < k$.

Note that from (4') it follows that o_2 is the only point above the line through $p_{i,i}$ and $p_{i+1,i+1}$, there-

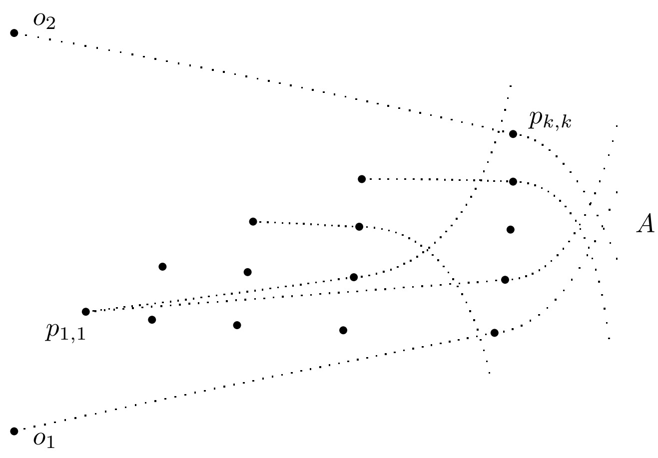

Figure 2: The Arrangement of Points. X-coordinates are not to scale, dotted lines represent straight lines.

fore, the edge joining $p_{i,i}$ and o_2 belongs to Π. Let D_i, $1 \le i \le k$, be the convex polygon in Π to the left of this edge when traversed from $p_{i,i}$ to o_2. Using (5) and (5') it is easy to prove the following proposition.

Proposition 5 *$D_l \neq C_{i,j}$ and $D_l \cap A = \emptyset$, for $1 \le l \le k$, $1 \le i \le j \le k$.*

Propositions 3 and 5 imply that $F_3(n) - n$ goes to infinity as n increases. For future reference, we compute the bound on F_3 obtained when $|A| = 1$ and $k = 3, 5, 6$.

Observation 1 $F_3(9) \ge 9$, $F_3(18) \ge 20$, $F_3(24) \ge 27$.

4 Generalization

Now we want to show that, for any $h \ge 3$, it is possible to put together h sets of $k(k+1)/2$ points in such a way that for each $t \in \{1, \ldots, h\}$, the set $\{p^t_{i,j}\}_{1 \le i \le j \le k}$ satisfies properties (1)-(5) and $\cup_{s \neq t}\{p^s_{i,j}\}$, playing the role of A, satisfies (3')-(5').

Let $h \ge 3$. Take a regular h-gon with vertices $q_1, \ldots, q_h$ (in clockwise order). Assume that q_1 and q_2 lie on the y-axis. Now construct the set $\{o_1, o_2\} \cup \{p_{i,j}\}_{1 \le i \le j \le k}$ from the previous setion taking $\{o_1, o_2\}$ to be $\{q_1, q_2\}$. In addition, require that the projection of every $p_{i,j}$ on the y-axis falls in the middle section of the segment $q_1 q_2$ when divided into three equal parts. Now compress the x-coordinates of the points $p_{i,j}$, i.e., multiply them by a constant, until they are so close to the segment $q_1 q_2$ that all the lines mentioned in properties (1)-(5) are so slanted that they intersect the contiguous sides $q_2 q_3$ and $q_h q_1$ in the third section adjacent to $q_1 q_2$. Call these points $\{p^1_{i,j}\}$ and do the same in each side $q_t q_{t+1}$ (where $q_{h+1} = q_1$) to obtain the points $\{p^t_{i,j}\}$.

Let $S_{h,k} = \{q_i\}_{1 \le i \le h} \cup \{p^t_{i,j}\}^{1 \le t \le h}_{1 \le i \le j \le k}$.

Given any convex partition of $S_{h,k}$, define $C^t_{i,j}$ and D^t_i as in the previous section, for each set $\{q_t, q_{t+1}\} \cup \{p^t_{i,j}\}_{1 \le i \le j \le k}$.

The construction of $S_{h,k}$ allows the set $\cup_{s \neq t}\{p^s_{i,j}\} \cup \{q_i\}_{i \neq t, t+1}$ to play the role of A in Propositions 4 and 5. Therefore, Propositions 3, 4 and 5 imply the following result.

Proposition 6 *For all $1 \le t_1, t_2 \le h$*

(i) $C^{t_1}_{i_1,j_1} \neq C^{t_2}_{i_2,j_2}$ if $1 \le i_1 \le j_1 < k$, $1 \le i_2 \le j_2 < k$ and $(i_1, j_1, t_1) \neq (i_2, j_2, t_2)$.

(ii) $C^{t_1}_{i,j} \neq D^{t_2}_l$ if $1 \le i \le j \le k$ and $1 \le l \le k$.

(iii) $D^{t_1}_{l_1} \neq D^{t_2}_{l_2}$ if $1 \le l_1, l_2 \le k$ and $(l_1, t_1) \neq (l_2, t_2)$.

5 Analysis

Let $n_{h,k}$ be the number of points in $S_{h,k}$, so

$$n_{h,k} = h + h\frac{k(k+1)}{2} \qquad (1)$$

Take any convex partition of $S_{h,k}$. By the last proposition, the polygons $C^t_{i,j}$, D^t_i are all distinct if $j < k$. This adds up to $hk(k-1)/2 + hk = hk(k+1)/2$ faces. Now, each point $p^t_{i,k}$ must be joined to some point not in $\{q_t, q_{t+1}\} \cup \{p^t_{i,j}\}$, because by property (2) there is a line through $p^t_{i,k}$ that contains no point of this set. Pick an edge going from $r = p^t_{i,k}$ to a point s in $(\{q_{t'}, q_{t'+1}\} \setminus \{q_t, q_{t+1}\}) \cup \{p^{t'}_{i,j}\}$, $t' \neq t$, and let E_{rs} be the convex set to the left of this edge oriented from r to s if $t < t'$ and form s to r in the other case. Clearly, each E_{rs} is different from every C and D polygon that we counted above and from every $E_{r's'}$ polygon associated with other edge $r's' \neq rs$. Moreover, each rs edge appears at most twice, so we have at least $hk/2$ additional polygons in the convex partition. Hence,

$$G(S_{h,k}) \ge \frac{hk(k+1)}{2} + \frac{hk}{2} = n_{h,k} - h + \frac{hk}{2} \qquad (2)$$

Now we think of h as fixed and write n_k in place of $n_{h,k}$. Solving for k in (1) and substituting in (2) gives

$$\begin{aligned} G(S_{h,k}) &= n_k - h + \frac{h}{2}\left(\frac{-1 + \sqrt{1 + \frac{8(n_k - h)}{h}}}{2}\right) \\ &> n_k - \frac{5}{4}h + \sqrt{\frac{(n_k - h)h}{2}} \end{aligned}$$

Hence, $F_h(n_k) > n_k - \frac{5}{4}h + \sqrt{\frac{(n_k-h)h}{2}}$. Now we derive a formula valid for every n.

Proposition 7 *$F_h(n) > n - \frac{15}{8}h + \sqrt{\frac{(n-h)h}{2}}$, for $n \ge h$.*

Proof. Let k be an integer such that $n_k \le n < n_{k+1}$. By Proposition 1, $F_h(n) \ge F_h(n_k) + n - n_k > n - \frac{5}{4}h + \sqrt{\frac{(n_k-h)h}{2}}$. Now, $\sqrt{(n-h)h/2} - \sqrt{(n_k-h)h/2} < \sqrt{(n_{k+1}-h)h/2} - \sqrt{(n_k-h)h/2} = h\sqrt{k+1}/(\sqrt{k+2} + \sqrt{k})$. But $\sqrt{x+1}/(\sqrt{x+2} + \sqrt{x})$ decreases quickly to $1/2$ and for $x = 1$ its value is less than $5/8$, so the result follows for $k \ge 1$. The case $k = 0$ can be verified separately, using that in this situation $h \le n < 2h$ and $F_h(n) \ge (3/2)(n-h) + 1$. □

The best lower bound for $F(n)$ that can be obtained by means of the sets $S_{h,k}$ is attained when $k = 5$ and h varies. Since $n_{h,5} = 16h$, using formula (2) we get $G(S_{h,5}) \ge (35/32)n_{h,5} > (12/11)n_{h,5}$, $h \ge 3$. Therefore, taking into account the monotonicity, we get

Proposition 8 $F(n) > \frac{12}{11}n - 2$, *for* $n \geq 4$.

Proof. If $n \geq 48$, say $16h \leq n < 16(h+1)$, use the fact $F(16h) > (12/11)16h$ and Corollary 2. If $4 \leq n < 48$, the result follows from Corollary 2, the obvious result $F(4) = 3$ and Observation 1. □

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