Reconfiguring planar dihedral chains

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Abstract

We consider the dihedral model of motion for chains with fixed edge lengths, in which the angle between every pair of successive edges remains fixed. A chain is *flat-state connected* if every planar configuration can be transformed to any other via a series of dihedral motions which maintain simplicity. Here we prove that three classes of chains are flat-state connected. The first class is that of chains with unit-length edges and all angles in the range $(60°, 150°)$. The second is the class of chains for which a planar monotone configuration exists. The third class includes, but is not limited to, chains for which every angle is in the range $(\delta, 2\delta)$, for $\delta \leq \frac{\pi}{3}$.

1 Introduction

The dihedral model for three-dimensional linkages resembles (and is designed to approximate) the "ball and stick" molecular model, used in introductory chemistry courses. Edges have fixed lengths and are not allowed to intersect. The angle between any two edges with a common vertex must also remain fixed. Thus any linkage motion can be decomposed into basic dihedral motions. A basic dihedral motion is defined on a selected edge of a given linkage. The entire linkage on one side of the selected edge is rotated rigidly about the axis of the edge (see Figure 1). This maintains all angles fixed.

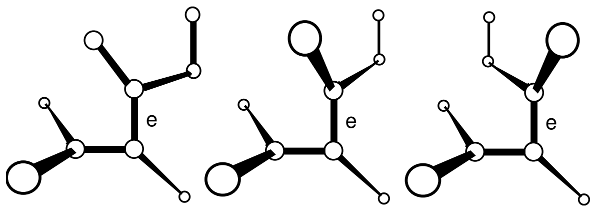

Figure 1: Three snapshots of a dihedral motion about edge e.

Soss and Toussaint [6] showed that deciding whether a chain can be flattened is NP-hard. They also developed a quadratic time algorithm to determine if the dihedral rotation about one edge results in edge-crossings. They gave a lower bound of $\Omega(n \log n)$, and this was nearly matched in the special case where the rotation is a full revolution. Soss, Erickson and Overmars [4] showed that pre-processing hardly helps if a series of such rotation queries is to be made. Several results on dihedral reconfigurations appear in the doctoral thesis by Soss [5]. A main open problem remaining from that research is to determine whether all planar (flat) configurations of a given chain are connected by a series of dihedral motions. This has come to be known as the *flat-state connectivity problem*, and accordingly when the answer is positive we say that a chain is "flat-state connected". One reason that planar configurations have received attention is that they may be useful as an intermediate (canonical) form during the reconfiguration of three-dimensional linkages. Demaine, Langerman and O'Rourke [3] considered chains with non-acute angles that can be "produced" through the apex of a cone, in a first geometric attempt to model a "protein machine". It was shown that chains are producible if and only if they can be flattened. Problems regarding flat-state connectivity were solved in [1, 2], by imposing additional restrictions to polygons, chains and trees. For example, orthogonal graphs were shown to be flat-state *disconnected*, as well as other linkages which were partially rigid. Open chains with non-acute or equal angles are flat-state connected, as are orthogonal polygons with unit edges. Of particular relevance to this paper is the class of open chains with unit-length edges and all angles in the range $(60°, 90°)$, shown to be flat-state connected in [2]. Here, we significantly expand this range of angles and prove that two new classes of open chains are flat-state connected.

In the following sections, we say that a chain consists of vertices $v_0, \ldots, v_n$ and edges $e_0, \ldots, e_{n-1}$. The edge e_i connects vertices v_i and v_{i+1}. Let α_i denote the angle at v_i, between e_{i-1} and e_i [1].

2 Chains which have a monotone configuration

Theorem 1 *Open chains that have a strictly monotone flat embedding are flat-state connected.*

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[1]Not to be mistaken with the *turning angle* of the chain. Here we have $(0 < \alpha_i < \pi)$. $\alpha_i = \pi$ would mean that the two edges could be considered as one and v_i could be ignored.

Proof. (sketch) Let D be a flat strictly monotone embedding of an open chain, i.e. any line parallel to the y-axis intersects D at a single point. Let C be another flat embedding of the same chain. We will prove that C can be reconfigured to D using dihedral motions. This implies the theorem, since D serves as a canonical configuration.

Let C be embedded in some plane Q. Let l be a line that lies in Q. Let P be a half plane bounded by l, contained above Q, as shown in Figure 2(a).

At the start of iteration i, $e_0, e_1, \ldots, e_{i-1}$ lie in P and conform to the layout of these edges in D, i.e. any line in P parallel to l intersects D at most once. The edges $e_i, e_{i+1}, \ldots, e_{n-1}$ lie in Q and these edges have the layout of C. The angle between P and Q is almost zero. Either e_i has to rotate around the edge e_{i-1} by an angle of almost π, or it is already nearly in the correct position. In the latter case all we need to do is lift v_i, l and P a little out of Q until e_i lies in the plane containing P. The edge e_i and the chain in Q have to be rotated slightly for this motion to be possible.

So assume that e_i is not in its correct position . Let e'_i denote the location where it has to move to. There are three cases to be considered. Either e_i and e'_i lie on the same side of l, they lie on opposite sides of l, or e_i lies on l. Since D is strictly monotone e'_i cannot lie on l. The first two cases are illustrated in Figure 2(b) and (c).

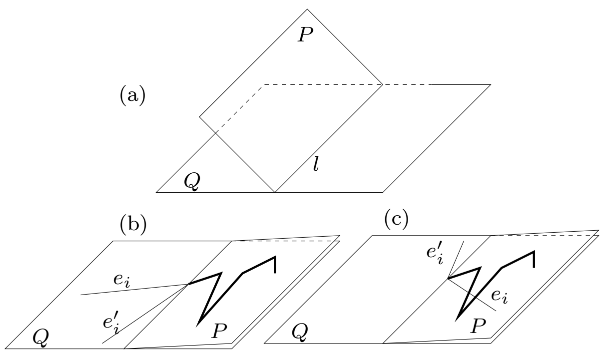

Figure 2: (a) Planes P and Q; (b) e_i and e'_i on the same side of l; (c) e_i and e'_i on opposite sides l

The theorem can be proved by examining the possible values of α_i. Let ϕ be the smallest angle between e_{i-1} and l. Let θ be the smallest angle between e_{i-1} and plane Q so $0 < \theta \leq \phi \leq \pi/2$.

We first observe that $\alpha_i > \phi$ since D is strictly monotone. If e_i and e'_i lie on the same side of l we have $\alpha_i > \pi - \phi$. This condition implies that $\alpha_i > \pi/2$. We rotate P around l until θ is equal to $\pi - \alpha_i$. This happens before P is vertical. During the rotation we have $0 < \theta \leq \pi - \alpha_i < \phi$. For each value of θ we can rotate C_0 around v_i so that the angle between e_{i-1} and e_i falls in the range $[\theta, \pi - \theta]$. This range contains α_i since $\alpha_i \leq \pi - \theta$ and $\theta < \pi/2 < \alpha_i$. So we can maintain an angle of α_i between e_{i-1} and e_i by rotating C_0 in plane Q around v_i. Similarly we can show that the layout after iteration i can be moved to the same intermediate configuration. This is sufficient to prove this case.

If e_i and e'_i lie on different sides of l we have $\alpha_i < \pi - \phi$. We will show that we can rotate P, while e_i stays on the same side of l. We first increase the angle between P and plane Q from ϵ to $\pi/2$. The range $[\theta, \pi - \theta]$ contains α_i since $\alpha_i < \pi - \phi \leq \pi - \theta$. So we can maintain an angle of α_i between e_{i-1} and e_i. Since the largest angle between l and e_{i-1} remains constant at value $\pi - \phi$ and is greater than α_i, e_i will not cross line l. Using a similar reasoning we can show that we can push P down until it lies on the other side of l, while e_i remains on the same side of l.

Finally if e_i lies on l we have $\phi < \pi/2$ and $\alpha_i = \pi - \phi$. First rotate P into a vertical position, during which rotation e_i does not move. We then rotate P left or right; during this rotation we move e_i until e_i reaches the position of e'_i. We can maintain an angle of α_i between e_{i-1} and e_i since $\pi/2 < \alpha_i = \pi - \phi \leq \pi - \theta$ implies that the range $[\theta, \pi - \theta]$ contains α_i. □

3 Chains with local angle restrictions

In this section, we show that if some simple relations between adjacent angles hold, a chain is flat-state connected. Define $\beta_i = \min(\alpha_i, \pi - \alpha_i)$.

Theorem 2 *Chains with angles α_i such that $\alpha_i \leq \beta_{i-1} + \beta_{i+1}$ or $\alpha_i \geq \pi - \max(\beta_{i-1}, \beta_{i+1}) + \min(\beta_{i-1}, \beta_{i+1})$ for $2 \leq i \leq n-2$ are flat-state connected.*

Proof. (sketch) Let C and D be two flat configurations of a chain that satisfies the conditions of the lemma. Assume without loss of generality that C is embedded in a plane Q. Let P be a plane that is parallel to and lies above plane Q at distance ϵ.

At the start of iteration i edges $e_0, e_1, \ldots, e_{i-1}$ lie in P. The edges $e_{i+1}, e_{i+2}, \ldots, e_{n-1}$ lie in Q. Edge e_i connects v_i in P to v_{i+1} in Q. The edges $e_0, e_1, \ldots, e_{i-1}$ conform to the layout of D, the edges $e_i, e_{i+1}, e_{i+2}, \ldots, e_{n-1}$ have the layout of C.

For all i let γ_i be the variable angle between e_i and P with $\gamma_i \leq \pi/2$. We move P upward, $e_0, e_i, \ldots, e_{i-2}$ remain in P, e_{i-1} drops below P, e_i stays above Q and $e_{i+1}, e_{i+2}, \ldots, e_{n-1}$ remain in Q. The motion continues, with angles γ_{i-1} and γ_i increasing, until e_{i-1} and e_i lie in a plane perpendicular to P. For an illustration, see Figure 3.

The motion of e_{i-1} can then be reversed, while e_i moves to its correct position. In other words, the layout before and after iteration i can be moved to the same intermediate configuration, that in which

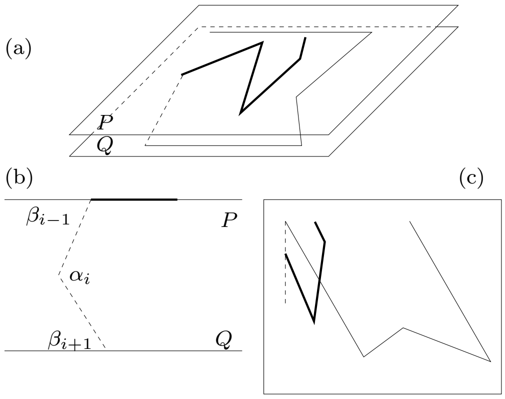

Figure 3: Edges in P are bold; edges in Q are solid; edges between P and Q are dashed. (a) planes P and Q; (b) side view when e_{i-1} and e_i lie in plane perpendicular to P and Q; (c) top view of the same configuration.

e_{i-1} and e_i lie in the perpendicular plane. This is sufficient to prove the theorem.

We first assume that $\beta_{i-1} \geq \beta_{i+1}$. If $\alpha_i \leq \beta_{i-1} + \beta_{i+1}$ let g be such that $\alpha_i = g(\beta_{i-1}+\beta_{i+1})$. If $\alpha_i \geq \pi - (\beta_{i-1}-\beta_{i+1})$ let g be such that $\alpha_i = \pi - g(\beta_{i-1}-\beta_{i+1})$. So $0 < g \leq 1$.

Suppose we have rotated the edge e_{i-1} out of the plane P such that $\gamma_{i-1} = f\beta_{i-1}$ with $0 < f \leq g$. Suppose we also have rotated e_i with $\gamma_i = f\beta_{i+1}$. It is not hard to show that when $f = g$ we have reached the situation where e_{i-1} and e_i lie in a plane perpendicular to P. Also it can be shown that for any value of f we can maintain the angles α_{i-1} and α_{i+1} by rotating the chains in P and Q. What remains to be shown is that we can maintain the angle α_i.

Below we show that α_i falls in the range $[f(\beta_{i-1} + \beta_{i+1}), \pi - f(\beta_{i-1} - \beta_{i+1})]$ for all values of f. This implies that we can maintain the angle α_i at v_i by rotating e_i and Q around v_i .

If $\alpha_i = g(\beta_{i-1} + \beta_{i+1})$, then $\alpha_i \geq f(\beta_{i-1} + \beta_{i+1})$. Since $g\beta_{i-1} \leq \pi - g\beta_{i-1}$ we have $\alpha_i \leq \pi - g\beta_{i-1} + g\beta_{i+1} \leq \pi - f(\beta_{i-1} - g\beta_{i+1})$. If $\alpha_i = \pi - (g\beta_{i-1} - \beta_{i+1})$, then $\alpha_i \geq g\beta_{i-1} + g\beta_{i+1} \geq f(\beta_{i-1} + \beta_{i+1})$. Also $\alpha_i \leq \pi - f(\beta_{i-1} - g\beta_{i+1})$.

To complete the proof we have to consider that case that $\beta_{i-1} < \beta_{i+1}$. However this case is similar to the case that $\beta_{i-1} \geq \beta_{i+1}$. So the theorem holds. □

The above theorem implies that any chain with $\delta \leq \alpha_i \leq 2\delta$ is flat state connected for any value of δ with $0 < \delta \leq \pi/3$.

4 Unit length chains

In this section we show that unit length chains with all dihedral angles in the range $(60°, 150°)$ are flat-state connected. Proofs for some of our claims are left out due to space constraints.

In a plane perpendicular to the original, we use a canonical configuration, defined as follows: the first edge v_1v_2 of the given chain must point up[2] . From there, each successive edge v_iv_{i+1} is placed so that v_{i+1} reaches a position with maximum height (without interfering with edges already fixed in place).

First we prove that a canonical chain is simple. Let the notation $v_a > v_b$ denote that v_a is higher than v_b in the vertical plane. We can show that in a canonical chain no edge can point down at a slope greater than $30°$ from horizontal, and two successive edges cannot both point down. Also, if v_iv_{i+1} points up, then v_{i+2} is at least half a unit higher than v_i. These two claims lead to the following two results:

Lemma 3 *Once a particular height h is reached by the canonical chain, the remaining chain cannot reach more than half a unit below h.*

Corollary 4 *If e_1 and e_2 are consecutive edges that point up, no successor of e_2 can intersect e_1.*

Lemma 5 *A canonical configuration has the property that every third vertex has monotonically increasing height. In addition, if edge e_i points up, then v_{i+3} is at least $\frac{1}{2}$ higher than v_i.*

Proof. Consider any three consecutive edges, e_1, e_2, e_3. We will show that v_4 must always be higher than v_1. If none of the three edges point down, then the claim holds trivially.

Suppose that e_2 points up. We know that a possible height decrease due to e_1 is less than $\frac{1}{2}$, and that e_2 and e_3 combine to a height increase of at least $\frac{1}{2}$.

Instead, if e_2 points down, then the other two edges point up. Thus e_1 and e_2 increase height by at least $\frac{1}{2}$, and this increase cannot be negated by e_3.

If the first edge points up, then only one edge can point down so the latter can be combined with its predecessor for a net height increase of at least $\frac{1}{2}$. □

Lemma 6 *Every six consecutive vertices result in a height increase of at least $\frac{1}{2}$.*

Proof. Let the six edges be $e_1, \ldots, e_6$. If e_1 or e_4 point up then by Lemma 5 there is a triplet of consecutive edges ($e_1e_2e_3$ or $e_4e_5e_6$) that gains $\frac{1}{2}$ in height. The other triplet does not lose height, so the claim is true. If both e_1 and e_4 point down, then e_2, e_3 and e_5 must point up . e_6 may point up or down. Thus pairs (e_3e_4) and (e_5e_6) each contribute a height increase of at least $\frac{1}{2}$. e_2 contributes positively, and e_1 can lose at most $\frac{1}{2}$. □

[2]We say that an edge v_iv_{i+1} *points down* if v_{i+1} is strictly lower than v_i. Otherwise the edge *points up*. Pointing left and right are defined similarly, with vertical edges symbolically defined to be pointing right.

Lemmas 3 and 6 imply that edge e_{i+6} and its successors cannot intersect e_i or its preceding edges. Thus what remains is to show that no six consecutive edges in canonical form can self-intersect.

From the angular restrictions of the problem definition, we know that no three consecutive edges can intersect. We can prove that no four or five consecutive edges in canonical form intersect. The proofs are similar to the following:

Lemma 7 *Six consecutive edges in canonical form cannot intersect.*

Proof. We know that five consecutive edges do not intersect in canonical form. Thus we focus on proving that e_1 does not intersect e_6.

Case 1: e_1 points up. By corollary 4, e_2 must point down if there is to be an intersection. Thus e_3 points up. If e_6 points down (implying e_5 points up) we have $v_6 > v_7 > v_5 > v_2 > v_1$ so we are done. If instead e_6 points up, we must look at e_5: if it points up, we have $v_7 > v_6 > v_5 > v_2 > v_1$, implying no intersection. Thus the only dangerous configuration remaining has e_5 pointing down (and e_4 pointing up). We know that v_6 is at least $\frac{1}{2}$ higher than v_4, which is no more than $\frac{1}{2}$ lower than v_2. So $v_7 > v_6 > v_2 > v_1$.

Case 2: e_1 points down (so e_2 points up). By Lemma 5, $v_7 > v_4 > v_1$. Since $v_1 > v_2$, we must only prove that $v_6 > v_1$. Clearly we only worry if e_6 points up. Let us examine the pair e_4e_5. If e_4 points up, this would mean that $v_6 > v_4$, which proves our claim. So instead assume that e_4 points down, which means e_3 and e_5 point up. Since v_5 is at least $\frac{1}{2}$ higher than v_3, and v_3 is no more than $\frac{1}{2}$ lower than v_1, we have $v_6 > v_5 > v_1$. □

This concludes the proof of the following theorem:

Theorem 8 *A chain that is in canonical form must be simple.*

Theorem 9 *Any two planar chains with edges of unit length and angles in the range* $(60°, 150°)$ *are flat-state connected.*

Proof. Let the given chain be in the horizontal plane. We begin by lifting the first edge so that it projects vertically onto the second edge. Now suppose that we have part of the linkage still in the original configuration, and part of it has been lifted into a vertical plane and is in canonical form. We want to move the canonical portion of the chain into a position above the next edge of the horizontal portion, as demonstrated in Figure 4. On the left of the figure is a partially lifted chain. The result will be a configuration such as the one on the right. Two simultaneous dihedral motions are performed during this operation.

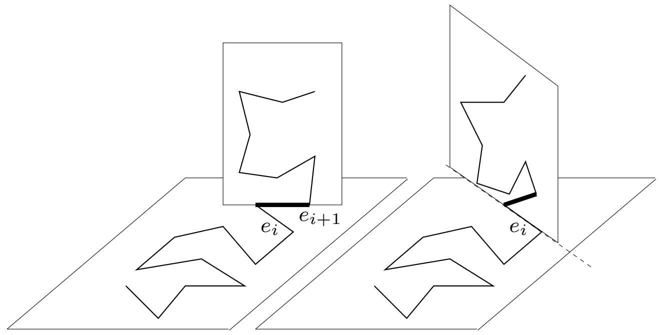

Figure 4: Lifting the next edge into canonical form.

Edges that are already in canonical configuration remain coplanar (in a vertical plane) throughout these motions. To do this, we rotate e_{i+1} about e_i and at the same time we rotate the canonical plane so that it always projects vertically through e_i. We call these two dihedral motions *primary*. We only need to intervene if an edge u points directly *up* (becomes vertical) during the primary motion. At this instant the chain C_u above u may be placed arbitrarily in either of two possible positions in the canonical plane. If the overall motion is to continue, u and the edge above it will no longer satisfy the greedy property. Thus we rotate C_u about u and proceed with the primary motion until it is complete or another edge becomes vertical. □

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