The Rotation Graph of k-ary Trees is Hamiltonian *

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Abstract

In this paper we show that the graph of k-ary trees, connected by rotations, contains a Hamilton cycle. Our proof is constructive and thus provides a cyclic Gray code for k-ary trees. Furthermore, we identify a basic building block of this graph as the 1-skeleton of the polytopal complex dual to the lower faces of a certain cyclic polytope.

1 Introduction

A k-ary tree is a rooted plane tree where each vertex has either k children or no children. Let $R(m,k)$ denote the graph whose vertices are all k-ary trees with m internal nodes, and where two trees are adjacent if they differ by a rotation, defined below. For $k = 2$, Lucas [6] and later Hurtado and Noy [2] showed that the rotation graph of binary trees is Hamiltonian. There exists a well known isomorphism [11] between the flip graph of triangulations of a convex polygon [2] and the rotation graph of binary trees. This isomorphism generalises to an isomorphism between graphs whose node sets are dissections of a convex polygon into m k-gons and $(k-1)$-ary trees with m internal nodes, respectively. See Figures 1, 2 and 3. In fact, a rotation for k-ary trees can be defined using this isomorphism to dissections of a convex polygon. A more direct definition of a rotation given by Sagan [9] is recalled in the next section. Another definition of rotations from Korsh [3] does not carry over to the flip graph of dissections, meaning that rotations for k-ary trees do not always correspond to edge-flips in the graph of dissections.

As our main results, we present a cyclic Gray code [10] for $R(m,k)$ and identify the graph $C(m-1,k)$ of weak compositions of $m-1$ into k parts as a basic building block of $R(m,k)$. Moreover, $C(m-1,k)$ turns out to be the 1-skeleton of the polytopal complex dual to the lower faces of a certain cyclic polytope.

There exist several Gray codes for k-ary trees. It is common to encode trees as a sequence of numbers, and then give a Gray code for these sequences; see [4, 1, 13]. Trees having a similar representation by numbers can have a very different natural structure. Our Gray code guarantees that adjacent trees in the list are very similar. To our knowledge no Gray code for k-ary trees based on rotations was known so far. Sagan [9] already proved that the flip-graph of rotations for k-ary trees is connected. We state our main result which is a direct consequence of Theorem 9.

Theorem 1 *There exists a cyclic Gray code for k-ary trees with m internal nodes in which consecutive trees differ by a rotation.*

2 Rotations for k-ary trees and dissections of a convex polygon

Let $D(m,k)$ be the graph of dissections of a convex n-gon into m convex k-gons, for $k \geq 3$ and $m \geq 1$. For $D(m,k)$ to exist, it is necessary and sufficient that $n = m(k-2)+2$. Two dissections in $D(m,k)$ are connected by an arc if they differ in the placement of exactly one diagonal (a 'flip'). By [7], the number of vertices of $D(m,k)$ is $\frac{1}{m}\binom{(k-1)m}{m-1}$.

The flip graph of dissections $D(m,k)$ has already arisen in [12] in the guise of the dual graph of the simplicial complex $\Delta(m,k)$ of dissections of a polygon into m convex k-gons, whose facets are the dissections and the vertices the diagonals. Tzanaki [12] proves that $\Delta(m,k)$ is vertex-decomposable, therefore shellable, and in consequence homotopy equivalent to a wedge of spheres; in fact, to a wedge of $\frac{1}{m}\binom{m(k-2)}{m-1}$ spheres of dimension $m-2$. In particular, $\Delta(m,3)$ is the (polar of the) associahedron [5], but for $k \geq 4$ and $m \geq 2$ the complex $\Delta(m,k)$ is not even isomorphic to the boundary complex of a PL sphere, much less a polytope.

We recall the definition of a rotation (i.e., an edge of the graph $R(m,k)$) given by Sagan [9]. A subtree T_v of a plane k-ary tree T *generated* by a vertex v consists of v and all its descendants. If v is an internal vertex then we let $v_1, v_2, \ldots, v_k$ be its children listed left to right and let $T_{v_1}, T_{v_2}, \ldots, T_{v_k}$ denote the trees they generate, respectively. We focus on the child $x = v_i$ and consider the list $L(T_v, x)$ of pairwise disjoint subtrees

$$L(T_v,x) := T_{v_1}, T_{v_2}, \ldots, T_{v_{i-1}}, T_{x_1}, T_{x_2}, \ldots, T_{x_k},$$

$$T_{v_{i+1}}, T_{v_{i+2}}, \ldots, T_{v_k}$$

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listed left to right in the order in which they are encountered in T (i.e., in depth-first order). Then a tree $\overline{T}$ is a flip (i.e., a rotation) of T, if it is isomorphic to T outside of T_v and there is some child y of v such that $L(T_v, x)$ and $L(\overline{T_v}, y)$ are isomorphic as ordered lists of rooted plane k-ary trees.

The graphs $D(m, k+1)$ and $R(m, k)$ are isomorphic. Figures 2 and 3 illustrate the isomorphism between $D(3, 4)$ and $R(3, 3)$.

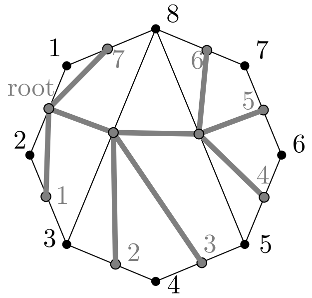

Figure 1: Bijection between a dissection of a convex polygon and a ternary tree.

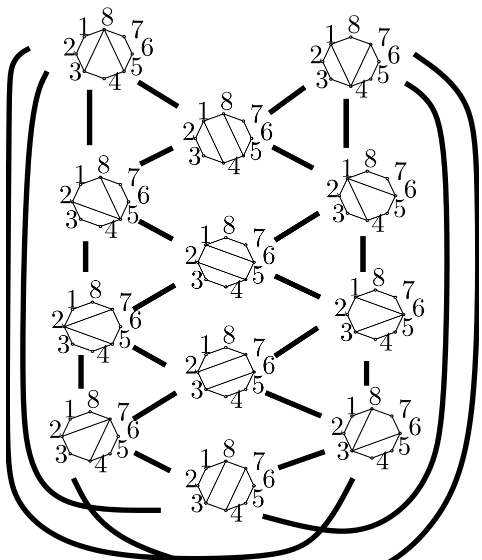

Figure 2: The graph $D(3, 4)$ is isomorphic to the graph $R(3, 3)$ of Figure 3.

3 Compositions

Let $r, s \geq 1$ be integers. A *(weak) composition of r into s parts* is an ordered s-tuple $(a_1, a_2, \ldots, a_s)$ of non-negative integers such that $a_1 + a_2 + \cdots + a_s = r$. We make the set $C(r, s)$ of all compositions of r into s parts into a graph by declaring two of them to be adjacent if they differ by one in exactly two positions that are connected by a (perhaps empty) sequence of 0's. For example, the composition $(1, 0, 2, 4, 0, 1)$ is adjacent to $(1, 0, 2, 3, 0, 2)$, but not to $(0, 0, 2, 4, 0, 2)$.

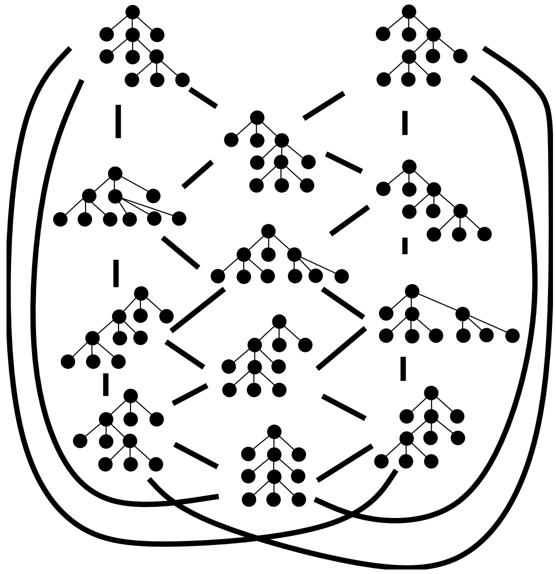

Figure 3: The graph $R(3, 3)$ is isomorphic to the graph $D(3, 4)$ of Figure 2.

Proposition 2 *The graph $C(r, s)$ is isomorphic to the dual graph of the simplicial complex of lower faces of the d-dimensional cyclic polytope $C_d(r+d)$ on $r+d$ vertices, where $d = 2s - 2$.*

Proof. The two sets have the same cardinality, because by Gale's Evenness Criterion (see, e.g., [14]) the number of lower facets of $C_d(r+d)$ is $\binom{r+d-d/2}{d/2} = \binom{r+s-1}{s-1} = \#C(r, s)$. Next, we define an injective map from $C(r, s)$ to the set of (indices of vertices contained in) lower facets of $C_d(r+d)$. To a weak composition $r = a_1 + \cdots + a_s$, associate the subset of $\{1, 2, \ldots, r+d\}$ of size d consisting of $s-1$ pairs of consecutive integers surrounded by s "holes", where the i-th "hole" has size a_i. For example, if $r = 3$ and $s = 4$, the composition $3 = 0 + 2 + 1 + 0$ corresponds to the set $\{1, 2, 5, 6, 8, 9\}$, which has "holes" $\emptyset$, $\{3, 4\}$, $\{7\}$, $\emptyset$, and indexes a lower facet of $C_6(9)$. This map is well-defined by Gale's Evenness Criterion, and injective by construction. Moreover, if two compositions are adjacent in $C(r, s)$, then exactly two "holes" in the corresponding facets of $C_d(r+d)$ differ in size by exactly ± 1, respectively ∓ 1. The two facets therefore share $d-1$ vertices, and are thus adjacent. □

Remark 3 *The faces of dimension at most $s-1$ of the complex dual to the lower faces of $C_d(r+d)$ form a polyhedral subdivision of the r times dilated $(s-1)$-dimensional standard simplex $r\Delta^{s-1}$, where Δ^{s-1} is the convex hull of the unit vectors in $\mathbf{R}^s$. The graph*

$C(r, s)$ is the 1-skeleton of this polyhedral decomposition, and the vertices of $C(r, s)$ are exactly the integer points of $r\Delta^{s-1} \subset \mathbf{R}^s$. Their coordinates correspond to the compositions of r into s parts (cf. Figure 4).

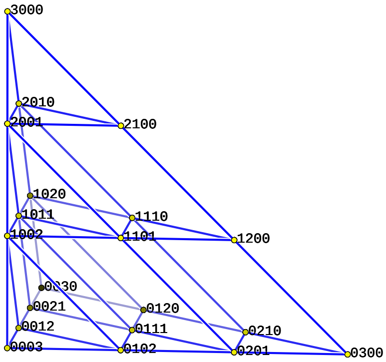

Figure 4: The graph $C(3, 4)$ viewed as the 1-skeleton of a subdivision of a 3-dimensional simplex.

Proposition 4 (see, e.g., [8]) *For each graph $C(r, s)$ with $r \geq s \geq 1$, there exists a Gray code $L(r, s)$ with endpoints $(r, 0, \dots, 0)$ and $(0, \dots, 0, r)$.*

Proof. For $r \geq 1$, set $L(r, 1) = r$. For $r \geq s > 1$, $L(r, s)$ is given by

$$L(r, s-1) \circ 0,\ \overline{L(r-1, s-1)} \circ 1,\ L(r-2, s-1) \circ 2,$$
$$\overline{L(r-3, s-1)} \circ 3, \dots,\ L(0, s-1) \circ r,$$

where $\overline{L(i, s-1)}$ is the list $L(i, s-1)$ in reverse order. □

For example, the Gray code $L(3, 3)$ for $C(3, 3)$ is

(300), (210), (120), (030), (021), (111), (201), (102), (012), (003).

4 A hierarchy for dissections

In [2] a hierarchy for triangulations was defined: all triangulations of convex polygons with any number of vertices are organized as nodes of a certain (infinite) tree. We generalise this approach and arrange all dissections of n-gons into k-gons into another rooted infinite tree T_k, which will be defined in the following by assigning a unique parent to every dissection but one. The vertices of each $D(m, k)$ are defined on polygons having $n = m(k-2) + 2$ vertices, labelled in counterclockwise order. These vertices will lie on the level m of T_k, so that the root of T_k, namely a k-gon, is at level 1. Thus, the dissections on level $m+1$ in this tree are defined on polygons having $n + k - 2$ vertices $\{p_1, \dots, p_{n+k-2}\}$.

For $m \geq 1$ let $V = \{p_n, p_{n+1}, \dots, p_{n+k-2}\}$ be the set of "last" vertices of the polygons in $D(m+1, k)$. Fix a dissection $\delta_* \in D(m+1, k)$, and for all $p_{n+i} \in V$, let a_i be the number of diagonals incident to p_{n+i} in δ_*. The total number of diagonals incident to vertices of δ_* in V will be called $\ell = a_0 + \dots + a_{k-2}$, so that the ordered tuple of non-negative integers $\mathbf{a}(\delta_*) = (a_0, \dots, a_{k-2}) \in C(\ell, k-1)$ is a composition of ℓ into $k-1$ parts.

Definition 1 *The parent $\delta \in D(m, k)$ of a dissection $\delta_* \in D(m+1, k)$ is the dissection obtained by removing all ℓ diagonals in δ_* incident to vertices in V, placing ℓ new diagonals incident to p_n into the "hole" created, and removing the $k-2$ vertices in $V \setminus \{p_n\}$. The* children *of a dissection $\delta \in D(m, k)$ are all dissections in $D(m+1, k)$ whose parent is δ. A general child of δ will be denoted δ_*. The* first child *δ_f of δ is the one associated to the composition $\mathbf{a}(\delta_f) = (\ell, 0, \dots, 0)$; the* last child *$\delta_l$ is associated to $\mathbf{a}(\delta_l) = (0, \dots, 0, \ell)$.*

Remark 5 *If there are $\ell - 1$ diagonals incident to the last vertex p_n of δ, then the children of δ are obtained by adding the $k-2$ vertices in $V \setminus \{p_n\}$ to δ, and distributing the new total of ℓ diagonals incident to p_n among all vertices in V. In particular, the first child δ_f can be equivalently defined by placing a new k-gonal tile on top of the edge $p_1 p_n$ and leaving all diagonals incident to p_n. Similarly, the last child δ_l is obtained from δ by placing a new tile on top of the edge $p_{n-1} p_n$ (thereby relabelling p_n to p_{n+k-2}). Note that the reason that there are now ℓ diagonals incident to p_n is that the former edge $p_1 p_n$ of the convex hull is now a diagonal. Figure 5 shows all children of a dissection $\delta \in D(4, 4)$.*

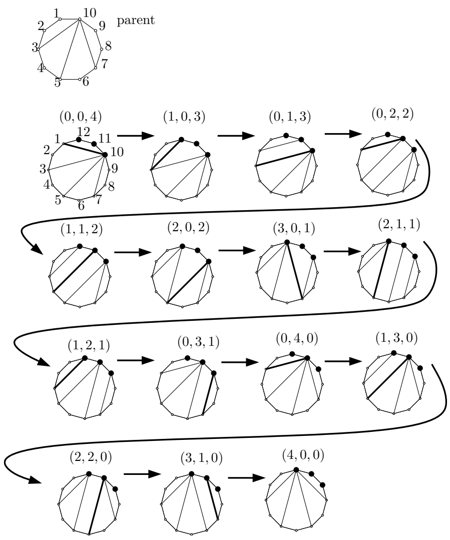

Figure 5: A Hamilton path for the children of a dissection according to a Gray code for compositions.

Remark 6 *Because $D(m, k)$ inherits the symmetry of the n-gon it contains several copies of each compo-*

sition graph $C(\ell, k-1)$, where ℓ is the total number of diagonals incident to some choice V of "last" vertices of the n-gon. Moreover, because each $C(\ell, k-1)$ is isomorphic to the dual graph of the complex of lower facets of the cyclic polytope $C_{2k-4}(\ell + 2k - 4)$ by Proposition 2, the graph $C(\ell_1, k-1)$ is an induced subgraph of $C(\ell_2, k-1)$ whenever $\ell_1 \leq \ell_2$.

Lemma 7 *Let δ be any vertex of $D(m, k)$ and let $\ell-1$ be the number of diagonals incident to the vertex p_n of the dissection δ. Then $C(\ell, k-1)$ is isomorphic to the subgraph of $D(m+1, k)$ induced by the children of δ.*

Proof. We constructed the children of δ by distributing the diagonals which are incident to p_n in δ among V. Every way of distributing the diagonals yields a unique child, and hence a unique composition. Thus, there is a bijection between the vertex sets. It is straightforward that an edge of the graph $C(\ell, k-1)$ corresponds to flipping one of the diagonals incident to a vertex of V, such that it remains incident to V. On the other hand, flipping such a diagonal away from V yields a dissection δ_x which has a different parent. Since the number of diagonals incident to V decreases, the composition assigned to δ_x does not belong to $C(\ell, k-1)$. Any two children of δ differ by edges incident to V, thus they are not adjacent by flipping an edge which is not incident to V. □

5 A Hamilton cycle

In the following adjacency between two dissections is denoted with '$\sim$'.

Lemma 8 *Let dissections δ, δ^1 and δ^2 of $D(m, k)$ be given.*
Then $\delta^1 \sim \delta^2$ implies $\delta^1_f \sim \delta^2_f$ and $\delta^1_l \sim \delta^2_l$. Moreover, there exists a Hamilton path formed by the children of δ whose endpoints are δ_f and δ_l.

Proof. The first statement follows immediately from Remark 5. The Hamilton path for the children of δ follows from Lemma 7 and Proposition 4. The endpoints of this path are the compositions $(0, \ldots, 0, \ell)$ and $(\ell, 0, \ldots, 0)$, where $\ell - 1$ is the number of diagonals incident to p_n of δ. These two compositions correspond to the dissections δ_f and δ_l. □

Figure 5 shows a Hamilton path among the children of a dissection and the corresponding Gray code for compositions. The diagonal which is exchanged in each step is drawn in bold.

Theorem 9 *The graph $D(m, k)$ contains a Hamilton cycle for all $k \geq 3, m \geq 1$, with the exception of $D(2, 3)$.*

The proof of Theorem 9 is based on Lemma 8. We omit the details in this abstract.

6 Acknowledgements

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