Cover Contact Graphs

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Abstract

We study properties of the cover contact graphs (CCG). These graphs are defined by a pair $G = (S, C)$, where S is a set of objects (called seeds) in the plane, and C is a collection of discs or triangles (called covers) covering the seeds, with the property that the interior of those discs are mutually disjoint. The contact graph of that cover set is a CCG. Then we study three basic properties of CCGs taking into account the nature of the seeds and of the covers. Namely, whether there exists a connected CCG on a fixed seed set. Whether is it possible to realize a given graph as a CCG, and finally we try to enumerate certain classes of CCGs.

1 Introduction

In two disciplines very much related between them as Computational Geometry and Graph Drawing there exists a very broad literature on covering problems and intersection graphs. In the first case, we try to cover some objects with covers attending some properties. And in the second case, the vertex set of a graph is a collection of objects and the edges represent their intersections. Thus is not surprising that sometimes both problems appear at the same time and then we want to cover a given collection of objects and to consider the intersection graph of the covers. This is our case. A first example of this kind of problems appears in Koebe's Theorem (rediscovered many times. In this sense, check [10] for a very interesting report on Koebe's theorem and its interconnections) that establishes that any plane graph can be realized as a coin graph. When in a coin graph we cover a point set in the plane by discs centered at the points, the discs must have disjoint interiors, therefore the intersection is just a tangent contact. In this paper, we study the same class of graphs but we do not assume that the discs are centered at the points. Changing this condition, the properties of the graphs so obtained also change a lot as we will see later. Additionally, we cover other objects as discs and triangles by discs and triangles respectively and we see that again a very different behavior can be observed (some other similar representations have been considered previously, for instance in [3] not only vertices are represented by discs but also the faces).

More formally, as it is established in the abstract above, a cover contact graphs (CCG) is defined by a pair $G = (S, C)$, where S is a set of objects (called seeds) in the plane, and C is a collection of discs or isothetic triangles (called covers) covering the seeds, with the property that the interior of those discs (or triangles) are mutually disjoint. The contact graph of that cover set is a CCG. In other words, we join two seeds by an edge if their covers are tangent (or their borders have a non-empty intersection). An example of a CCG is depicted in Figure 1.

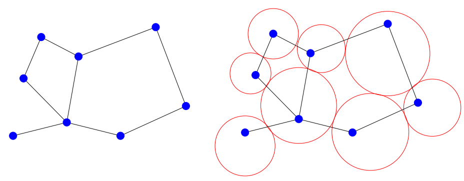

Figure 1: Realization of a graph

Not surprising, changing the nature of the seeds and of the covers different properties are obtained. We study here three of the more basic problems. Namely, *(1)* Whether there exists a connected CCG on a fixed seed set. *(2)* Whether is it possible to realize a given graph as a CCG. *(3)* We try to enumerate certain classes of CCGs. The first question is inspired by a result that says that given a point set on the plane to decide if there exists a connected coin graph on it, is an NP-complete problem [1]. In fact, even to decide if some coins can cover a given set is an NP-complete problem [2]. On the other hand, the second and the third problems are key problems from the point of view of Graph Drawing and Combinatorics respectively.

2 The seeds are points

As it is said in the Introduction, we make the first distinction attending the nature of the seeds, and, of course, the first class of seeds to be considered is that of points.

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2.1 Points in the plane

In this subsection, the seeds will be always points in the plane and the covers will be either discs containing the points, or equilateral triangles with an edge parallel to the abscissae axis and the seed must be its bottom vertex.

Firstly, regarding connectivity, it is easy to see that, on the opposite of what happened for coin graphs, for any seed set it is always possible to find a connected CCG on it.

Proposition 1 *For any seed set, there exists always a connected CCG on it (for both cases of covers, discs and triangles). Furthermore, it is possible to construct one of them in $O(n \log n)$ (being n the number of points). Even more, for any seed set, there exists always 2–connected CCGs on it (for both cases of covers, discs and triangles). Furthermore, it is possible to construct one of them in $O(n^2)$ (being n the number of points).*

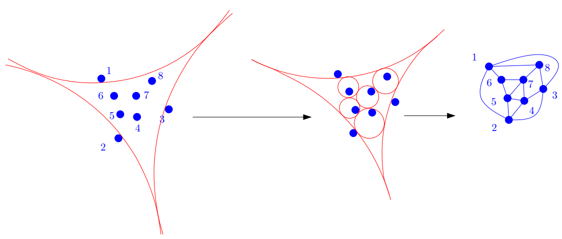

Figure 2: A 2-connected CCG

Focussing on realization, although we have seen that to find a connected CCG is always possible if the seeds are points in the plane, we will see now that to realize a given graph is much more difficult. Of course, if we do not fix the seeds, given a planar graph G, Koebe's theorem [9] guaranties that we can find a seed set S such that it is possible to realize G on S. But if the seeds are fixed previously, then the problem turns to be a very difficult one.

Theorem 2 *Given a point set S in the plane, and a planar graph G, to decide whether there exists a CCG on S isomorphic to G is an NP-complete problem.*

Nevertheless in some cases it is possible to say something more. The next result give some necessary conditions on a graph to be realizable. In order to establish that result we define a graph on a seed set S inspired in the Sphere of influence graph defined by Toussaint [11] (see also [6, 7] for more results in the sphere of influence graphs). Given a point p of a seed set S, we associate to p the union $C(p)$ of all the maximal empty (in the sense that do not contain any seed in their interior) circles centered at vertices of its Voronoi region, the intersection graph of the sets so defined will be called the hyperinfluence graph of S (denoted $HI(S)$).

Proposition 3 *Let G be a graph realizable as a CCG on a seed set S. Then* (1) *G is a subgraph of $HI(S)$;* (2) *it is possible to give a plane representation of G with S as its vertex set and each edge with at most two rectilinear segments (one bend per edge).*

The first condition of Proposition 3, says us that graphs as that depicted in Figure 3 cannot be realizable as a CCG. And it is worthy to mention that the second condition will be used later to characterize CCGs when the seeds lie on a line.

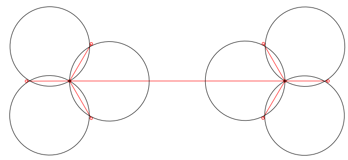

Figure 3: A non–realizable graph

To the light of Figure 3 one can ask what is the minimum number of seeds with a non-realizable graph. In this way we can enunciate

Proposition 4 *Any graph with 4 or less vertices is representable as a CCG on any seed set.*

On the other hand, it is possible to give a collection of six points in convex position, such that its Delaunay triangulation is not representable as a CCG, see Figure 4.

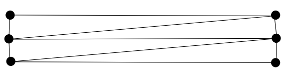

Figure 4:

Finally, regarding enumeration, as any planar graph can be realized by Koebe's theorem the remarkable result obtained in [4] holds in this case, namely, there exists $g \cdot n^{-7/2}\gamma^n n!$ realizable graphs of n vertices, where $g \approx 0.4970043999 \cdot 10^{-5}$ and $\gamma \approx 27.2268777685$ are constants, given by explicit analytic expressions.

2.2 Points on a line

If we restrict the points to be on a line, then we can be more precise in the results obtaining a more accurate idea of what can be done and what cannot be done. In fact, we introduce an interesting particular case of CCGs. If we denote by $\Re^2_+$ to the half plane defined by the points with non-negative ordinate, we call a CCG$^+$ graph to a pair $G = (S, C)$, where S is a set of objects (called seeds) in $\Re^2_+$ such that each seed has at least one point on the line $y = 0$, and C is a collection of discs or isothetic triangles (called covers) in $\Re^2_+$ covering the seeds, with the property that the interior of those discs (or triangles) are mutually disjoint. The contact graph of that cover set is a CCG$^+$. (Figure 5) shows a CCG$^+$ graph.

We will follow the same structure as in the previous subsection.

To achieve the connectivity is an easy task when the points lie on a line as we see in the next result.

Proposition 5 *Let S be a set of n seeds on a straight line, then* (1) *there exists always a realizable (as a CCG) C_n (cycle of length n) on it;* (2) *there exists always a tree on S realizable as CCG$^+$.*

About realization, we have seen in Proposition 5 that on any seed set, C_n is always realizable. One can ask if Koebe's theorem is still valid when the seeds lie on a line, but this is not true since in [8] is proved that there is a plane triangulated graph with only 12 vertices such that for every placement of the vertices on a straight line at least one edge must bend at least twice in the resulting drawing.

This result together with Proposition 3 imply that the plane graph described in that paper is not realizable as a CCG if the seeds lie on a straight line. And, it is not difficult to see that any tree is realizable.

Proposition 6 *Given any tree T, it is possible to choose seeds in a line such that T is realizable as a CCG$^+$ on that seed set.*

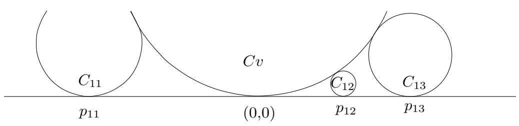

Figure 5: First step in realization of a tree T.

In Proposition 6 the seeds are not fixed on the line, an step beyond this situation is when the seeds are not fixed but their ordering on the line. In this case, not every tree is realizable as a CCG.

Given a labeled tree T, and an ordering $\mathcal{S}$ of the vertices of the tree, we say that T is realizable on $\mathcal{S}$ if there exist points on a line such that T realizable on those points and the ordering of the vertices of the tree on the line is $\mathcal{S}$.

A subgraph as that depicted in Figure 6 is called a forbidden chain. More precisely, given a graph $G = (V, E)$ an edge $e \in E$, a *forbidden chain for* e is a matching $M = (V', E')$ of G described as follows:

1. The vertex set V' is a sequence of points in $\Re$, $A_1 < A_2 < B_1 < A_3 < B_2 < A_4 < \cdots < A_{2n-1} < B_{2n-2} < A_{2n} < B_{2n-1} < B_{2n}$

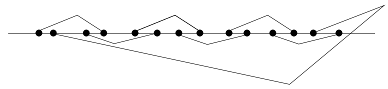

Figure 6: A forbidden chain.

2. For any $i \in \{1, 2, \ldots, 2n\}$, $\{A_i, B_i\} \in E'$.
3. $e = \{E_1, E_2\}$ has its end points separated by M like one of these cases:
   - $e_1 \in (-\infty, A_2)$ and $e_2 \in (A_{2n}, B_{2n-1})$
   - $e_1 \in (-\infty, A_1)$ and $e_2 \in (A_2, B_1)$
   - $e_1 \in (A_2, B_1)$ and $e_2 \in (B_{2n-1}, +\infty)$
   - $e_1 \in (A_1, A_2)$ and $e_2 \in (B_1, B_2)$
   - $e_1 \in (A_2, B_1)$ and $e_2 \in (B_2, +\infty)$
   - $e_1 \in (-\infty, A_{2n-1})$ and $e_2 \in (A_{2n}, B_{2n-1})$

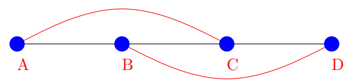

Figure 7: T is not realizable on S. $\{A, C\}, \{B, D\}$ is an elemental forbidden chain

Theorem 7 *Let T be a labeled tree, and $\mathcal{S}$ and ordering of its vertices. Then, the following conditions are equivalent:* (1) *T is realizable on $\mathcal{S}$ as a CCG;* (2) *it is possible to draw T with its vertices on a line with only one bend per edge and such that the ordering of those vertices is $\mathcal{S}$;* (3) *there are no forbidden chains.*

As far as the third condition can be checked in linear time, we obtain

Corollary 8 *Let T be a labeled tree, and $\mathcal{S}$ and ordering of its vertices. Then, it can be decided in linear time whether T is realizable on $\mathcal{S}$ as a CCG or not.*

A similar result can be establish regarding CCG$^+$.

Theorem 9 *Let T be a labeled tree, and S and ordering of its vertices. Then, the following conditions are equivalent:* (1) *T is realizable on S as a CCG$^+$;* (2) *it is possible to draw T in $\Re^2_+$ with its vertices on a line with only one bend per edge and such that the ordering of those vertices is S;* (3) *there are no elemental forbidden chains.*

Corollary 10 *Let T be a labeled tree, and $\mathcal{S}$ and ordering of its vertices. Then, it can be decided in linear time whether T is realizable on $\mathcal{S}$ as a CCG$^+$ or not.*

So, once we have established results about realizability firstly without fixing the seeds, and then fixing the order in which they appear, we can give a result that describes how are all the CCG$^+$'s when the covers are triangles.

Proposition 11 *Given a seed set S on the line, and covers triangles isothetic to the triangle with vertices in the points $(0,0)$, $(-1,1)$, and $(1,1)$. Then, any T realizable as a CCG$^+$ on it can be obtained following the next recursive method. Starting with L=S and while $|L| > 1$, choose the closest pair (u,v) of L as an edge of T and delete from L either u or v.*

Although the result of Proposition 11 is stated for a very particular case of triangles, it can be extended easily to any kind of triangles, changing the metric between the points.

The keys to enumeration are some results on realization. Namely, Propositions 6 and 11 and Theorem 9. Thus we can establish

Theorem 12 *The following results hold for seeds on a line:* (1) *The number of labeled trees with n vertices realizable as a CCG (or CCG$^+$) is n^{n-2} (all the labeled trees);* (2) *given an ordering $\mathcal{S}$ of the natural numbers from 1 to n. The number of labeled trees realizable as a CCG$^+$ on $\mathcal{S}$ is the Catalan number C_{n-1});* (3) *given a seed set S on a line. The number of labeled trees realizable as a CCG$^+$ on S is 2^{n-2}.*

3 The seeds are discs or triangles

In this section, we consider either discs or isothetic triangles in the plane as a seed set, and we will cover them with the same kind of objects, this is to say, the covers for discs are discs and the covers for triangles are isothetics triangles.

Regarding connectivity, we have

Proposition 13 *If the seeds are triangles with its bottom vertex in a horizontal line. Then there exists always a connected CCG$^+$ on it.*

We cannot extend the result of Proposition 13 any further. So in the case of triangles we can give collections of seeds such that no 2-connected CCG can be constructed on it (see Figure 8 (a)). And, in the case of discs, collections of seeds such that no connected CCG$^+$ can be constructed on it, as Figure 8 (b) shows. Even, it is not difficult to translate the example of Figure 8 (b) to CCGs in general; in order to get this goal it suffices to consider a huge disc just under all the other discs.

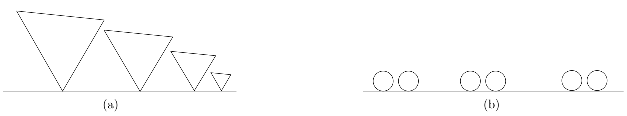

Figure 8: This seeds cannot be covered by a connected CCG$^+$.

Focussing on realization and enumeration, it is trivial to see that the result of Theorem 2 is still valid if we consider as discs as seeds instead of points (in order to see this remark it suffices to substitute the points by small discs). In fact, the results of Proposition 3 are also easily adaptable to the case of a set of discs as the seed set.

Regarding enumeration, the results obtained for points are clearly upperbounds in the case of discs or triangles, and the exact enumeration of the CCG representable on a concrete seed set seems to be a very difficult task.

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