A binary labelling for plane Laman graphs and quadrangulations

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Abstract

We provide binary labellings for the angles of quadrangulations and plane Laman graphs which are in analogy with Schnyder labellings for triangulations [W. Schnyder, Proc. 1st ACM-SIAM Symposium on Discrete Algorithms, 1990].

1 Introduction

Schnyder-labellings are by now a classical tool to deal with planar graphs. One of their first applications was to obtain convex drawings of such graphs on a small grid [14].

A Schnyder-labelling is a special labelling of the angles of a plane graph with three colors. Schnyder [14] introduced this concept for triangulations, or maximal planar graphs. The angle-labelling corresponds directly to a decomposition of the edge-set into three spanning trees, or a *Schnyder-wood*. Felsner adapted this idea for 3-connected planar graphs [4].

Other classes of planar graphs, such as maximal bipartite planar graphs and *planar Laman graphs*, admit a decomposition of the edge set into two trees. Our motivation for this work was to obtain a binary labelling for these classes of graphs analogous to Schnyder's.

Let us recall that a graph is *planar* if it can be embedded in the plane; a *plane graph* has already been embedded in the plane [7]. We remark that for a binary labelling of a plane graph G, it is not necessary that G is embedded with straight-line edges. We use only the combinatorics of the embedding, i.e. incidences of vertices, edges and faces, for our proofs.

A *quadrangulation* is a 2-connected plane graph where each interior face has four edges. A quadrangulation Q is a maximal bipartite plane graph if and only if the outer face of Q has four edges.

A tree decomposition for maximal bipartite planar graphs has been obtained by several authors [13, 12, 6, 1, 10]. The binary labelling "inherits" the tree decomposition property from Schnyder's labelling. More precisely, we obtain that every quadrangulation can be decomposed into two spanning trees by duplicating edges of the outer face.

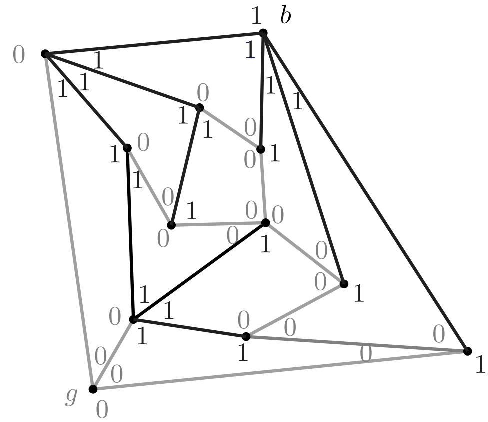

Figure 1: A binary labelling for a quadrangulation.

Fraysseix and Mendez [5] relate Schnyder labellings for triangulations to 3-orientations and shelling orders. They also consider "separating decompositions" for maximal bipartite planar graphs which are closely related to binary labellings.

We show a binary labelling for plane Laman graphs. Laman graphs [9] are well known in the context of rigidity theory. A Laman graph on n vertices contains $2n-3$ edges and every subgraph which is induced by k vertices contains at most $2k-3$ edges.

Every planar Laman graph can be embedded as a *pointed pseudo-triangulation*[8]. A vertex v of a plane straight-line graph is called *pointed* if it has an incident angle greater than π. A pointed pseudo-triangulation is a maximal pointed plane straight-line graph; this means that every vertex is pointed and adding any (non-crossing) edge yields a non-pointed vertex. In particular, the binary labelling holds for pointed pseudo-triangulations.

Other labellings for planar graphs have been investigated. Haas et al. [8], also see [11], defined "combinatorial pseudo-triangulations". Souvaine and Tóth [15] defined a vertex-face assignment for plane graphs.

Quadrangulations and plane Laman graphs are structurally different, because quadrangulations are bipartite graphs, whereas every pointed pseudo-

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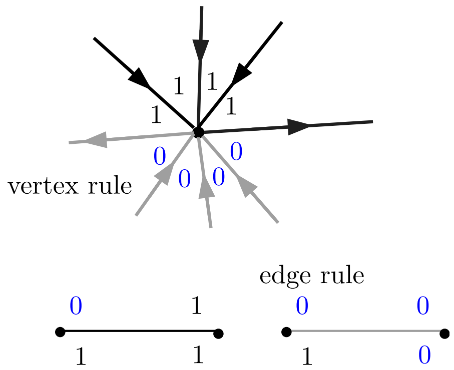

Figure 2: The vertex rule and the edge rule of a binary labelling.

triangulation contains a triangle. However, adding a (non-crossing) edge to a maximal bipartite plane graph yields a plane Laman graph, which in turn can be embedded as a pointed pseudo-triangulation. Thus, we believe that many concepts for pseudo-triangulations apply to quadrangulations and vice versa. The binary labelling only represents one aspect of this interesting fact. Another example is that every maximal bipartite planar graph can be obtained via a "Henneberg construction" [17] and also via "vertex splitting" [3]. As a last step of the construction one edge has to be deleted.

2 The binary labelling

Let a plane graph G be given. A *binary labelling* for G is a mapping from the angles of G to the set $\{0, 1\}$ which satisfies the following conditions:

1. There are two special vertices g and b on the outer face of G. All angles incident to g are labelled 0, all angles incident to b are labelled 1.

2. **Vertex rule** For each vertex $v \notin \{b, g\}$, the incident labels form a non-empty interval of 1*s* and a non-empty interval of 0*s*.

3. **Face rule** For each face (including the outer face) its labels form a non-empty interval of 1*s* and a non-empty interval of 0*s*.

4. **Edge rule** For each edge its labels either are $0, 1 - 1, 1$ or $0, 1 - 0, 0$; see Figure 2.

Observation 1 *The labelling of the edges induces an orientation: every edge is oriented towards its endpoint* $0, 0$, *respectively* $1, 1$. *In a binary labelling every vertex but* $\{g, b\}$ *has outdegree two.*

Both quadrangulations and pseudo-triangulations admit a binary labelling. Special properties of this labelling for quadrangulations are explained in the next section.

3 The binary labelling for quadrangulations

Let a quadrangulation Q be given such that the outer face of Q contains four vertices. For a binary labelling of Q we also require the following properties:

5. Each face has two adjacent 0-labels and two adjacent 1-labels, i.e. it is labelled $(0, 0, 1, 1)$.

6. The edges incident to g are labelled $0, 1 - 0, 0$ where the 1 stands on the right side of the edge (seen from vertex g).

Figure 1 shows a binary labelling.

Theorem 1 *Every quadrangulation with four vertices on its outer face admits a binary labelling.*

Proof. We use induction on the number of vertices $|V|$ of a quadrangulation Q. If $|V| = 4$ then a binary labelling exists, as shown in Figure 3 (left). For the induction step we distinguish two cases.

First, assume that Q contains an interior vertex v of degree two. Removal of v and its two incident edges yields a quadrangulation Q'. By induction, Q' admits a binary labelling. Reinserting v and its incident edges into Q' maintains the binary labelling, as shown in Figure 3 (right).

We now assume that Q contains no interior vertex of degree two. In this case, there exists a face incident to the special vertex g which can be *contracted* towards g. A face q incident to g is contractible if it does not contain the other special vertex b. A contraction of $q = \{e', e, f, f'\}$, where $\{e', e, f, f'\}$ are the edges of q in cyclic order and e' and f' are incident to g and e and f are incident to the vertex p opposite to g, identifies e with e', f with f' and p with g. It can be interpreted as a continuous movement of p and its incident edges to g.

A contraction of q yields a quadrangulation which by induction admits a binary labelling. In particular, the edges e' and f' are labelled $0, 1 - 0, 0$ towards g (Property 6 of the binary labelling). Now, we reverse the contraction. This operation maintains the binary labelling outside of the face q. It remains to label the angles inside q. The vertex rule for the special vertex g requires that the angle incident to g is labelled with 0. Labelling the angle formed by e and e' with 1 maintains Property 6 for e' and guarantees the vertex rule for this vertex. Observe that now the edge e is labelled with endpoint $1, 1$. Hence, the angle at p inside q has to be labelled with 1 to guarantee the edge rule for e. Note that all other labels at p are 0. Thus, labelling p with 1 also ensures the vertex rule for p. Finally, labelling the angle formed by f and f' with 0 guarantees the edge rule for f and f'; here, we again use Property 6. Observe that the vertex rule is satisfied for this vertex; and q satisfies the face rule.

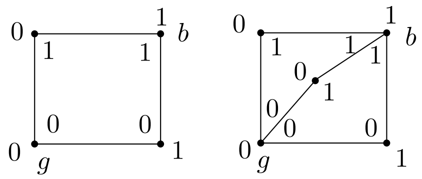

Figure 3: The basis of the induction and inserting a vertex of degree two.

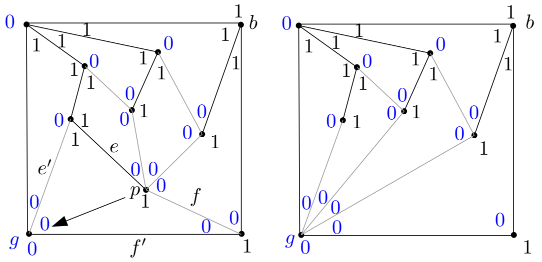

Figure 4: Contracting a quadrangle to the special vertex g.

Figure 4 shows a contraction of a quadrangle and the resulting binary labelling.

□

3.1 The two regions of a vertex

Analogous versions of the following results have been given by Schnyder [14] for triangulations and by Felsner [4] for 3-connected planar graphs. The proofs are omitted in this abstract.

We denote with T_0 the union of the edges which are oriented towards $0, 0$. We denote with T_1 the union of the edges which are oriented towards $1, 1$.

Lemma 2 *$T_i, i \in \{0, 1\}$, is a directed tree rooted at g, respectively b.*

Lemma 3 *There is no directed cycle in $T_0 \cup T_1^{-1}$. There is no directed cycle in $T_1 \cup T_0^{-1}$.*

For each interior vertex v and $i \in \{0, 1\}$, we define the i-path $P_i(v)$ starting at v as the path in T_i from v to the root of T_i. $P_0(v)$ and $P_1(v)$ have v as only common vertex. Therefore $P_0(v)$ and $P_1(v)$ divide the quadrangulation into two regions $R_0(v)$ and $R_1(v)$.

Lemma 4 *For any two distinct interior vertices u and v and for $i \in \{0, 1\}$ there holds the implication $u \in R_i(v) \Rightarrow R_i(u) \subset R_i(v)$.*

The following result has been proved with a different method in [1].

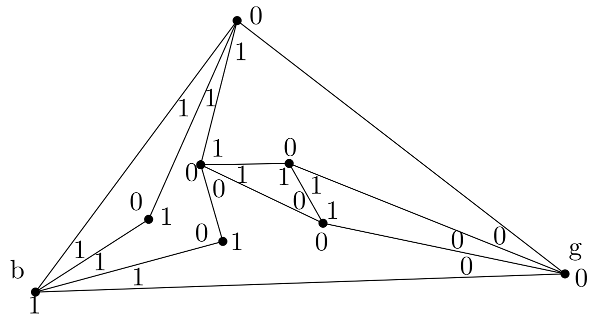

Figure 5: A binary labelling for a pointed pseudo-triangulation.

Lemma 5 *Every quadrangulation can be decomposed into two edge-disjoint spanning trees by duplicating edges of the outer face.*

Let Q be a quadrangulation and let $\widetilde{Q}$ be its dual graph. Then the "dual" of the two spanning trees T_i for $i = 1, 2$ are two spanning trees decomposing $\widetilde{Q}$.

4 A binary labelling for plane Laman graphs

We consider an analogous binary labelling for plane Laman graphs. Now, the special vertices b and g are adjacent convex hull vertices. Thus there is one edge which does not satisfy the edge rule.

Figure 5 shows a binary labelling for a pointed pseudo-triangulation.

Theorem 6 *Every plane Laman graph with three vertices on its outer face admits a binary labelling.*

Sketch of Proof. Our proof of the binary labelling is based on the *Henneberg construction*[17, 8, 16, 11]. A Laman graph can be constructed, starting from a triangle, by a sequence of vertex insertions of the following types (see Figure 6)

1. Add a degree-two vertex (Henneberg I step)
2. Place a vertex on an existing edge and connect it to a third vertex (Henneberg II step).

The binary labelling can be proved inductively. This amounts to showing that a Henneberg step maintains the labelling. To guarantee planarity, we make use of a lemma from Haas et al. [8]. We omit the details. □

It is well known that a Laman graph can be decomposed into two trees [17]. These trees can be obtained via the Henneberg construction, as indicated in Figure 6. The new vertex either is a leaf in both trees (Henneberg I step) or in one tree (Henneberg II step).

Although the binary labelling is based on the Henneberg construction too, it does not always give a decomposition of the graph into two trees; see Figure 7

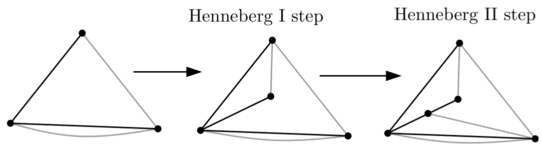

Figure 6: Constructing a decomposition into two spanning trees via Henneberg steps.

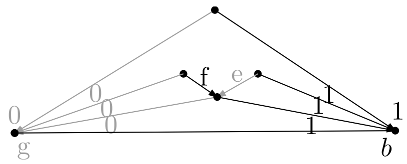

Figure 7: The binary labelling for plane Laman graphs does not induce a decomposition into two trees.

for a simple example: The angles around the special vertex g, respectively b, are labelled with 0, respectively 1. All edges incident to g are gray, all edges incident to b are black. Thus, if there is a decomposition of the edge set into two trees, then the edge f has to be black and oriented towards b, and the edge e has to be gray and oriented towards g. But then, the angle formed by e and f has to labelled with 1 and with 0, contradicting to properties of the binary labelling.

We remark that variants of the binary labelling hold for plane Laman graphs containing more than three vertices on the outer face.

5 Open problems

A main application of the Schnyder labelling for triangulations is a straight-line embedding of a triangulation on an $n-2$ by $n-2$ grid. Biedl and Brandenburg [2] recently showed that every planar bipartite graph has a straight-line embedding on a grid of size $\lfloor \frac{n}{2} \rfloor$ by $\lceil \frac{n}{2} \rceil - 1$. What is the corresponding grid size for planar Laman graphs? We did not succeed in applying the binary labelling. Another related question, posed by Haas et al. [8], is the following: Can every planar Laman graph be embedded as a pseudo-triangulation on a grid of small size?

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