Geometric realization of a projective triangulation with one face removed

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Abstract

Let M be a map on a surface F^2. A *geometric realization* of M is an embedding of F^2 into a Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of M is a flat polygon. In our talk, we shall prove that every triangulation G on the projective plane has a face f such that the triangulation of the Möbius band obtained from G by removing the interior of f has a geometric realization.

1 Introduction

A *triangulation* on a surface F^2 is a map on F^2 such that each face is bounded by a 3-cycle, where a k-cycle means a cycle of length k. We suppose that the graph of a map is always *simple*, i.e., with no multiple edges and no loops.

Let G be a map on a surface F^2. A *geometric realization* of G is an embedding of F^2 into a Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of G is a flat polygon. That is, a geometric realization of G is to express G as a polytope $P(G)$ in $\mathbb{R}^3$ such that $P(G)$ is homeomorphic to F^2, and that the 1-skeleton of $P(G)$ is homeomorphic to the graph of G. Note that we do not require the convexity of $P(G)$.

Steinitz's theorem states that a spherical map has a geometric realization if and only if its graph is 3-connected [7]. Moreover, Archdeacon et al. proved that every toroidal triangulation has a geometric realization [1]. In general, Grünbaum conjectured that every triangulation on any orientable closed surface has a geometric realization [5], but Bokowski et al. showed that a triangulation by the complete graph K_{12} with twelve vertices on the orientable closed surface of genus 6 has no geometric realization [3].

Let us consider nonorientable surfaces, in particular, the projective plane. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on the projective plane has a geometric realization. However, the surface obtained from the projective plane by removing a disk (i.e., a Möbius band) is embeddable in $\mathbb{R}^3$, and hence we can expect that a triangulation on the Möbius band has a geometric realization.

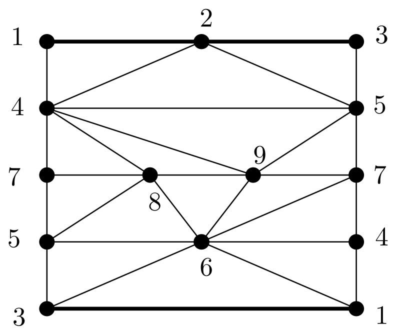

Figure 1: A Möbius triangulation with no geometric realization

For simple notations, we call a triangulation on the projective plane and that on the Möbius band a *projective triangulation* and a *Möbius triangulation*, respectively.

In the current work, we discuss geometric realizations of Möbius triangulations. However, Brehm [4] has already found a Möbius triangulation with no geometric realization, which is shown in Figure 1. (In Figure 1, identify vertices with the same label.) Can we get an affirmative result for geometric realizations of Möbius triangulations?

Let G be a projective triangulation and let f be a face of G. Let $G - f$ denote the Möbius triangulation obtained from G by removing the interior of f.

The following is our main theorem.

Theorem 1 *Every projective triangulation G has a face f such that the Möbius triangulation $G - f$ has a geometric realization.*

As far as we know, the current result seems to be the first affirmative result for geometric realizations of maps on nonorientable surfaces, since Brehm found the counterexample shown in Figure 1.

2 Sketch of the proof

In this section, we briefly explain our graph-theoretical proof of the theorem.

Let M be a map on a surface F^2 and let e be an edge of M. *Contraction* of e in M is to removed e and

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identify the two endpoints of e. (The inverse operation of contraction of an edge is called a *splitting of a vertex*.) If this yields a face bounded by a 2-cycle, then we replace the two parallel edges with a single edge. The contraction of an edge e is allowed only if the graph, say H, obtained from M by the contraction of e is simple. In this case, we say that e is *contractible*, and that M is *contractible* to H. We say that M is *irreducible* if M has no contractible edge.

Barnette [2] proved that the projective plane admits exactly two irreducible triangulations, which are the complete graph K_6 with six vertices and $K_4+\overline{K_3}$ (i.e., the quadrangulation by K_4 with each face subdivided by a single vertex), which are shown in the left-hand side in Figures 2 and 3, respectively. In particular, the latter contains a quadrangulation by K_4 with vertex set $\{1, 2, 3, 4\}$, called a K_4*-quadragulation*, which will play an important role in our proof.

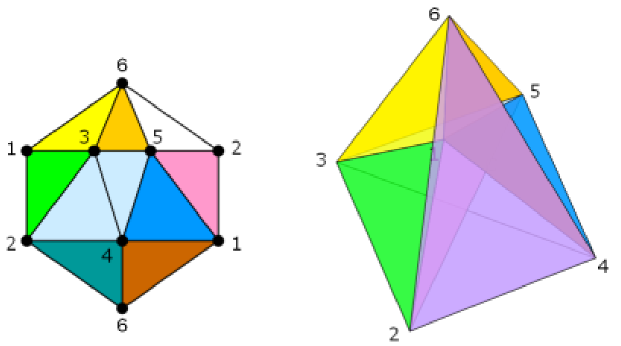

Figure 2: A geometric realization of K_6 minus face 256

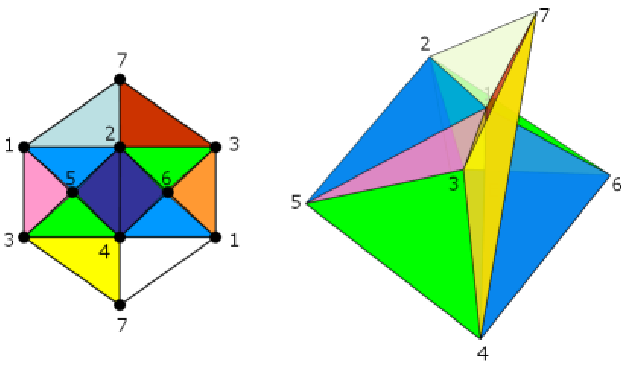

Figure 3: A geometric realization of $K_4 + \overline{K_2}$ minus face 147

Lemma 2 *Each of the two irreducible projective triangulation with one face removed has a geometric realization.*

The right-hand side of Figures 2 and 3 show geometric realizations of the two irreducible triangulations with one face removed, respectively. Note that each of the two triangulations is symmetric, the map with any one face removed has a geometric realization.

In order to prove Theorem 1, it is difficult to use induction on the number of vertices, though the first step of induction is verified by Lemma 1. Therefore, we introduce the following lemma to classify the set of all projective triangulations into two classes, which contain two irreducible projective triangulations independently.

Lemma 3 *Let G be a projective triangulation. Then G is contractible to K_6 if and only if G does not contain a K_4-quadrangulation.*

Throughout the proof, we use Menger's Theorem many times, which is well-known in graph theory and states that for a graph G and its two disjoint vertex-sets A, B with cardinality k, there are k disjoint paths joining A and B, unless G has a cut set X with cardinality at most $k-1$ separating A and B.

We first consider a projective triangulation G with a K_4-quadrangulation as a subgraph. Applying Menger's Theorem suitably, we can easily find a subdivision of $K_4+\overline{K_3}$. Since $K_4+\overline{K_3}$ itself satisfies Theorem 1 by Lemma 1, it is not difficult to construct a geometric realization of G with one face removed such that each path of G corresponding an edge of $K_4+\overline{K_3}$ is a straight-line segment, and that each 2-cell region of G corresponding a face of $K_4+\overline{K_3}$ is a flat triangle.

By Lemma 2, if a projective triangulation has no K_4-quadrangulationas a subgraph, then it is contractible to K_6. In other words, in this case, G has a K_6*-minor* as a subgraph, which is a map transformed into K_6 by a sequence of contracting and deleting edges.

Let K_5 denote a Möbius triangulation obtained from the triangulation K_6 by removing the 2-cell region consisting of five triangular faces incident to a single vertex. A K_5*-minor* is a map on the Möbius band transformed into K_5 by a sequence of contracting and deleting edges. Observe that there are two ways to split a vertex of K_5, depending on whether a new edge arisen by the splitting lies on the boundary of the Möbius band, or not. The former is called a *boundary splitting*, and the latter an *innr splitting*. Hence there are several homeomorphism-classes of the K_5-minors.

Lemma 4 *A K_5-minor has a geometric realization.*

For example, Figure 4 shows a K_5-minor obtained by five boundary splittings and its geometric realization, in which we can see that contracting v_i and v_i' for each i yields K_5.

Finally, we have to put in a 2-cell region, say R, with one face removed to the body of the geometric realization of a K_5-minor constructed in Lemma 4. By using Menger's Theorem carefully, we can take

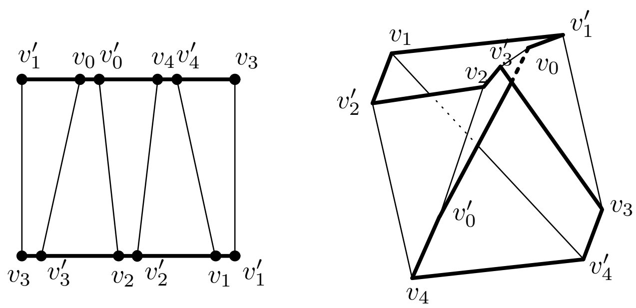

Figure 4: A K_5-minor obtained by boundary splittings of all five vertices and its geometric realization

an inner vertex v in R which has disjoint five paths $Q_0, \ldots, Q_4$ to five corners $v_0, \ldots, v_4$, as shown in Figure 5, for example. The argument is complicated, and hence we omit the details.

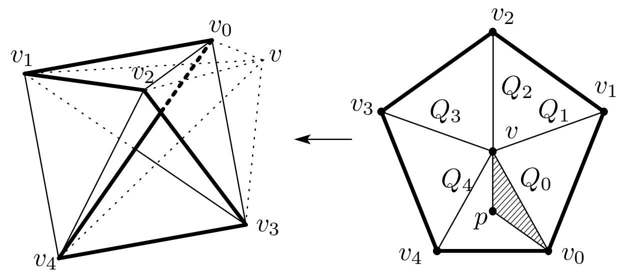

Figure 5: Putting in R with one face removed to K_5.

3 Conclusion and Conjecture

Our problem is as follows:

PROBLEM. Let M be a map on the projective plane and let f be a face of M. Does $M - f$ have a geometric realization?

Our main result is Theorem 1, that is, if M is a triangulation and f is some face of M, the answer for the above problem is "yes". Note that in Theorem 1, f cannot be chosen arbitrarily since there is a counterexample by Brehm shown in Figure 1. Moreover, we do not know whether the restriction to be a triangulation in Theorem 1 is actually needed. Therefore, it will be nice to consider the above problem when M is a Petersen graph on the projective plane with each face pentagon, which is a surface dual of K_6.

Why does the Brehm's counterexample have no geometric realization? A key of the proof is that in every spatial embedding of the map shown in Figure 1, the two disjoint 3-cycles 123 and 456 have a linking number at least 2. (See [6] for the definition of the linking number.) However, any two 3-cycles with straight segments have linking number at most one, a contradiction. Therefore, we have the following observation:

Observation 1 *If a Möbius triangulation G has a boundary cycle C of length 3 and a 3-cycle C' disjoint from C which forms an annular region with C', then G never has a geometric realization.*

A graph G is said to be *cyclically k-connected* if G has no separating set $S \subset V(G)$ of G with $|S| \leq k - 1$ such that each connected component of $G - S$ has a cycle. If we assume the cyclically 4-connectedness of a Möbius triangulation, we can avoid the situation described in Observation 1. So we conjecture the following, which will enable us to prove Theorem 1 easily.

Conjecture 1 *Let G be a projective triangulation. Then G is cyclically 4-connected if and only if $G - f$ has a geometric realization for any face f of G.*

Clearly, the above characterizes a geometrically realizable Möbius triangulation to be cyclically 4-connected. This answers the question given in [1].

In the previous section, we have briefly explained how to construct a geometric realization of a projective triangulation with one face removed. Of course, we needed an observation from a geometrical point of view. However, we feel that a graph-theoretical method is more essential, that is, we will need to find a geometrically realizable specific subgraph in an arbitrarily given projective triangulation.

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