Min-max-min Geometric Facility Location Problems

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Abstract

We propose algorithms for a special type of geometric facility location problem in which customers may choose not to use the facility. We minimize the maximum cost incurred to a customer, where the cost itself is a minimum between two costs, according to whether the facility is used or not. We therefore call this type of location problem a *min-max-min* geometric facility location problem. As a first example, we describe the Closer Post Office problem, a generalization of the minimum spanning circle problem. We show that this problem can be solved in $O(n)$ randomized expected time. We also show that the proposed algorithm solves two other min-max-min geometric facility location problems. One, which we call the Moving Walkway problem, seems to be the first instance of a facility location problem using time metrics.

1 Introduction

In this work we study facility location problems in which customers make some decision on whether they have some interest in using the facility or not. The facility is defined as a geometric object, and customers are not interested in using it if it is too far from their own location.

We assume there are n customers. If we denote by x the facility to be located, then $C_x(i)$ is the cost incurred to the ith customer if she uses the facility, and $C_{\bar{x}}(i)$ is the cost if she does not use the facility. A *min-max-min* facility location problem is a problem of the form:

$$\min_{x} \max_{1\leq i\leq n} \min\{C_x(i), C_{\bar{x}}(i)\}.$$

An interesting application of this model is transportation facility location. A transportation facility might be for instance a bus line, a subway station, or an air connection between two airports. When setting up a new facility of this type, a company must take into account its usefulness, since customers that already have access to a closer or faster existing transportation facility will certainly not use the new one.

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1.1 Related Works

The simplest form of facility location is maybe the Weber problem (see for instance [7], Chapter 1), in which a point minimizing the sum of distances to n other points is to be found. This problem dates back to the seventeenth century and many variations of it are still the subject of intense research nowadays.

Recently, Cabello et al. proposed algorithms for *reverse facility location* problems [3]. This term refers to *reverse nearest neighbor queries* in sets of points. Those queries receive a point as input and return the points from the set that have the query point as nearest neighbor. In reverse facility location, a new facility is located in such a way that the corresponding reverse nearest neighbor query returns the maximum number of points. The authors prove that this problem is 3SUM-hard and also propose algorithms for locating the facility with respect to additional min-max or max-min distance criteria. The problem we consider in Section 2 is a variant of this, which can be solved more efficiently. Reverse facility location can be considered as a discrete version of the problem studied in [5] of inserting, in a given set of points, a new point whose Voronoi cell has maximum size.

Several geometric optimization problems can be cast as finding minima on the upper envelope of a set of so-called *Voronoi surfaces*, generated by distance functions. In [8], it is proposed to construct completely this upper envelope to solve a family of problems, one of which is the minimum Haussdorff distance under translation between two point sets. General bounds on the description complexity of surface envelopes are given in Sharir and Agarwal's book on Davenport-Schinzel sequences [10]. They also propose algorithms for constructing the envelopes, which are used in the reverse facility location algorithms of [3]. The problems we consider in the following sections are all sufficiently simple so that we can avoid constructing the whole envelope of Voronoi surfaces.

In the min-max-min facility location problems we consider, the cost for a customer is the result of a (simple) minimization problem. If the facility is a transportation facility, then this minimum can be interpreted as a *time metric*. Time metrics have recently shown to be useful in the geometric analysis of transportation networks [1, 2]. The problem we consider in Section 4 is, to the authors' knowledge, the first instance of a facility location problem using time metrics.

1.2 Our Contributions

In Section 2, we define the Closer Post Office problem, a first, simple application of the proposed model. Suppose a new post office is to be installed in a city and we wish to minimize the maximum distance between any customer and the closest post office. If, for some customer, the new post office is not closer than the one she is used to go to, she will not use it and her cost will not vary. This is reminiscent of reverse facility location, except that the cost of all customers are taken into account, and not only the costs of the customers that actually use the facility. We observe that this problem boils down to finding a minimum height point on the upper envelope of a set of surfaces, each of which is the lower envelope of a horizontal plane and a cone.

In Section 3, we show that this problem is solvable in $O(n)$ randomized expected time.

In Section 4, we consider a transportation facility location problem, the Moving Walkway problem. It consists of finding the best location of a simple transportation facility, that is a moving walkway, modeled as an interval on the real line. Using the previous developments, we show that we can solve it in $O(n)$ randomized expected time as well.

In Section 5, we define a line location problem, the Highway problem. This problem can be interpreted as that of finding the optimal location of a highway, that customer might use to go quicker from point s_i to point t_i. Its interpretation in the geometric dual plane (mapping lines to points and points to lines) leads to a simple formulation similar to the previous ones.

Finally, Section 6 presents higher-dimensional generalizations of the previous problems.

2 The Closer Post Office problem

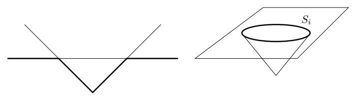
This problem is a generalization of the min-max center problem (or minimum spanning circle, see e.g. [9]) in which the distance function is the minimum between the actual distance to the facility and some constant.

Definition 1 (L_p-Closer Post Office) *Given n pairs (s_i, t_i) of points in the plane, find a point x^* which solves the following problem:*

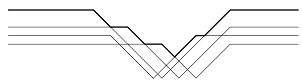
$$\min_{x\in\mathbb{R}^2} \max_{1\leq i\leq n} \min\{d(s_i,t_i), d(s_i,x)\},$$

where $d(.,.)$ is the L_p distance function.

We denote by $d_i(x) = \min\{d(s_i,t_i), d(s_i,x)\}$ the cost for customer i. The function $d(s_i,x)$ in the plane defines a cone centered on s_i. The function $d_i(x)$ therefore defines a surface f_i that is the lower envelope of the cone and the plane of equation $f(x) = d(s_i,t_i)$. We call S_i the region of equation $d(s_i,t_i) \geq d(s_i,x)$, which is a scaled and translated version of the unit disk for the L_p distance function that is used.



(a) A single surface f_i.



(b) The upper envelope of the surfaces f_i.

Figure 1: Illustration of the objective function yielded by the L_p-Closer Post Office problem.

Since we minimize the maximum distance $d_i(x)$, this amounts to finding the minimum height point on the upper envelope of the set of surfaces f_i. This is depicted schematically on Figure 1.

3 A General Algorithm

In order to solve the Closer Post Office problem, we first reformulate it in terms of the surfaces f_i. We let $h_i = d(s_i,t_i)$ be the height of the surface f_i. We define the disks $S_i(h)$ for each surface f_i and real number h as the regions of equation $d(s_i,x) \leq h$ if $h < h_i$ and $\mathbb{R}^2$ otherwise.

Lemma 1 *Solving the* Closer Post Office *problem is equivalent to finding the minimum value h^* of h for which the intersection $\bigcap_{i=1}^n S_i(h)$ is nonempty.*

We first solve the decision problem consisting of verifying whether $h > h^*$. This amounts to checking whether a set of disks has a nonempty intersection, which can be done in $O(n)$ randomized expected time using Seidel's algorithm (see e.g. [6]).

In order to solve the corresponding optimization problem within the same time bounds, we use Timothy Chan's reduction [4] of an optimization problem to its decision version. For that purpose, we have to decompose the original problem in m subproblems $\{Q_1, Q_2, \ldots, Q_m\}$ of size n/c for some constants m and c, such that the solution h^* of the original problem is the maximum of the solutions of the m subproblems.

We first make reasonable assumptions on the problem, namely that the surfaces f_i are symbolically tilted and the points s_i and t_i are in some general position so that the minimum h^* is on the intersection of a constant, say k, number of surfaces. To find a suitable subproblem decomposition, we partition the set of surfaces $\{f_i\}$ in $k+1$ disjoint subsets $F_1, F_2, \ldots, F_{k+1}$. Each subproblem Q_j is defined the same way as our original problem, but only on the union of k subsets F_ℓ of surfaces f_i. There are as many subproblems as there are k-subsets of the set $\{F_1, F_2, \ldots, F_{k+1}\}$, thus $m = k+1$. Each subproblem has size $\frac{k}{k+1}n$, thus we identify $c = (k+1)/k$. The minimum we look for is defined by k surfaces, and this k-tuple must appear in at least one of the subproblems Q_j. In all the other subproblems, the minimum cannot be smaller, hence the solution is given by the maximum of the solutions of the m subproblems. This gives us all necessary conditions for the reduction to work, and we can therefore solve the CLOSER POST OFFICE problem using Seidel's algorithm.

Theorem 2 *The L_p-CLOSER POST OFFICE problem can be be solved in $O(n)$ randomized expected time.*

Note that this algorithm can be applied to any similar problem, in which we look for the minimum on the upper envelope of a set of surfaces, each being a plane with a convex "hole" of constant description complexity.

4 The MOVING WALKWAY problem

The motivation for the next problem is the following. Suppose a new moving walkway is to be installed in a long corridor (for instance in the concourse of an airport). This walkway is to be used by people to go from one point of the corridor to another, and we wish to locate it so that it is the most useful. We model the corridor as the real line, source and destination points by pairs (s_i, t_i) of real numbers with $t_i \geq s_i$, and the moving walkway by an interval $[a, b]$ on the line, with $b \geq a$. We denote by $v^{-1} > 1$ the speed on the moving walkway, and assume that the speed outside the moving walkway is 1. Customers can only enter the walkway at point a and step down of it at point b. Our objective function is the maximum time needed to go from s_i to t_i.

The time to go from any point s of the line to any other point t is really a time metric [1, 2], defined as the minimum between the difference $t - s$ and the following sum:

$$|s-a| + |t-b| + v(b-a).$$

The first two terms are the time needed to go from s to the entrance of the walkway, and from the end of the walkway to t, respectively, while the third term is the time spent on the walkway. We can now define the problem.

Definition 2 (MOVING WALKWAY) *Given n pairs (s_i, t_i) of real numbers with $t_i > s_i$, and a real number $v \leq 1$, find a pair (a^*, b^*) which solves the following problem:*

$$\min_{(a,b)\in\mathbb{R}^2 : b \geq a} \; \max_{1\leq i\leq n} d_i(a,b)$$

where $d_i(a,b) = \min\{t_i - s_i, |s_i - a| + |t_i - b| + v(b-a)\}$.

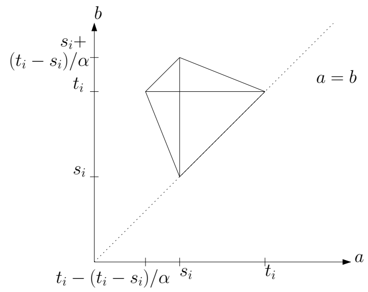


Figure 2: Edges of the surface defined by the function $d_i(a, b)$ in the MOVING WALKWAY problem.

The following lemma characterizes the situations in which customers decide not to use the walkway. We denote by $\ell(I)$ the length of an interval I.

Lemma 3 *The time metric $d_i(a,b)$ between s_i and t_i can be written as:*

$$d_i(a,b) = \begin{cases} t_i - s_i & \text{if } \ell([a,b]\cap[s_i,t_i]) \\ & \leq \alpha(b-a) \\ |s_i - a| + |t_i - b| & \\ +v(b-a) & \text{otherwise,} \end{cases}$$

where $\alpha = (v+1)/2$.

We now consider the function $d_i(a,b)$ in the plane (a,b). This function defines a surface f_i that has a structure similar to those appearing in the CLOSER POST OFFICE problem. The surface f_i is the lower envelope of a horizontal plane corresponding to the inequality $d_i(a,b) \leq t_i - s_i$, and a piecewise linear surface made of four patches. The vertices of f_i are the following: $d_i(s_i, t_i) = v(t_i - s_i)$, $d_i(t_i, t_i) = d_i(s_i, s_i) = t_i - s_i$, and $d_i(s_i, s_i + (t_i - s_i)/\alpha) = d_i(t_i - (t_i - s_i)/\alpha, t_i) = t_i - s_i$. The projection of the edges of the surface f_i on the plane (a,b) is depicted on Figure 2. If the speed on the walkway is infinite, that is when $v = 0$, then the surface f_i is the lower envelope of a horizontal plane and a rectilinear cone, as those appearing in the L_1-CLOSER POST OFFICE problem.

Lemma 4 *The MOVING WALKWAY problem with $v = 0$ is a special case of the L_1-CLOSER POST OFFICE problem.*

It can be checked that the algorithm described in the previous section applies to this problem, even when $v > 0$.

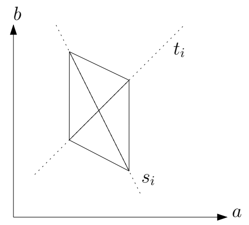


Figure 3: Edges of the surface defined by the function $d_i(a,b) = \min\{d(s_i,t_i), d_v(s_i,L) + d_v(L,t_i)\}$ in the HIGHWAY problem, where the line L has equation $y = ax + b$. The function has value 0 at the intersection of the lines defined by the points s_i and t_i in the dual plane, and $d(s_i,t_i)$ outside the parallelogram.

Theorem 5 *The* MOVING WALKWAY *problem can be be solved in $O(n)$ randomized expected time.*

5 The HIGHWAY problem

Our last problem is that of locating a line $L : y = ax+b$ in the plane. We denote by $d_v(x,L)$ the vertical distance between the point x and the line L.

Definition 3 (HIGHWAY) *Given n pairs (s_i,t_i) of points in the plane, find a line L^* which solves the following problem:*

$$\min_{L} \max_{1\leq i\leq n} \min\{d(s_i,t_i), d_v(s_i,L) + d_v(L,t_i)\}$$

This problem can be interpreted as that of locating a highway, which customers may use to go from point s_i to point t_i. When analyzed in the dual plane mapping lines to points and points to line, this problem can be shown to be very similar to the previous two ones, for it also consists of finding a minimum on the upper envelope of a set of surfaces, each of which is defined as the lower envelope of a plane and a cone, as depicted on Figure 3. The previous algorithm applies again in this situation.

Theorem 6 *The* HIGHWAY *problem can be be solved in $O(n)$ randomized expected time.*

6 Higher-dimensional variants

Clearly, the algorithm designed to solve the CLOSER POST OFFICE problem can also be used to solve the analogous problem in $k > 2$ dimensions instead of 2. The decision problem then consists of checking whether an intersection of k-dimensional balls is empty, which can be done in linear time as long as k remains constant. The 2-dimensional extension of the MOVING WALKWAY problem is of particular interest. It could model a situation in which for instance a new public transport connection is to be set up between two points. The problem is the following.

Definition 4 ($2D$-L_p-MOVING WALKWAY) *Given n pairs (s_i,t_i) of points in the plane, and a real number $v \leq 1$, find a pair (a^*,b^*) which solves the following problem:*

$$\min_{(a,b)\in\mathbb{R}^2\times\mathbb{R}^2} \max_{1\leq i\leq n} d_i(a,b)$$

where $d_i(a,b) = \min\{d(t_i,s_i), d(s_i,a) + d(t_i,b) + vd(a,b)\}$ and $d(.,.)$ is the L_p distance function.

In order for the algorithm to work, we have to make sure that the decision problem can be solved in linear time. We define the regions $S_i(h)$ as $\mathbb{R}^4$ if $d(s_i,t_i) \leq h$ and as

$$\{(a,b) \in \mathbb{R}^4 : d(s_i,a) + d(b,t_i) + vd(a,b) \leq h\}$$

otherwise. The decision problem amounts to checking whether the intersection of the regions $S_i(h)$ is empty. Those regions can be shown to be convex, hence the decision problem is a convex programming problem in 4 dimensions. This can be solved using a randomized algorithm in $O(n)$ time. From the previous developments, the optimization problem can be solved within the same time bound.

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