

# Maximizing the Guarded Interior of an Art Gallery\*

Ioannis Z. Emiris<sup>†</sup>Christodoulos Fragoudakis<sup>‡</sup>Euripides Markou<sup>§</sup>

## Abstract

In the Art Gallery problem a polygon is given and the goal is to place as few guards as possible so that the entire area of the polygon is covered. We address a closely related problem: how to place a fixed number of guards on the vertices or the edges of a simple polygon so that the total guarded area inside the polygon is maximized. Recall that an optimization problem is called APX-hard, if there exists an  $\epsilon > 0$  such that an approximation ratio of  $1 + \epsilon$  cannot be guaranteed by any polynomial time approximation algorithm, unless  $P = NP$ . We prove that our problem is APX-hard and we present a polynomial time algorithm achieving constant approximation ratio. Finally we extend our results for the case where the guards are required to cover valued items inside the polygon. The valued items or “treasures” are modeled as simple closed polygons.

## 1 Introduction

In the Art Gallery problem a polygon is given and the goal is to place as few guards as possible so that the entire area of the polygon is covered. Many variations of the Art Gallery problem have been studied during the last two decades ([11], [12], [13]). These can be classified with respect to where the guards are allowed to be placed (e.g. on vertices, edges, interior of the polygon) or whether only the boundary or all of the interior of the polygon needs to be guarded, etc. Most known variations are NP-hard. We address a closely related problem:

**Definition 1** *Given is a simple polygon  $P$  and an integer  $k > 0$ . The goal of the MAXIMUM AREA VERTEX GUARDS problem is to place  $k$  vertex guards so that the area of  $P$ 's interior that is overseen by the guards is maximum.*

The MINIMUM VERTEX GUARDS problem asks how to guard a polygon, with or without holes, us-

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<sup>†</sup>Department of Informatics and Telecommunications, National University of Athens, [emiris@di.uoa.gr](mailto:emiris@di.uoa.gr)

<sup>‡</sup>Computer Science, ECE, National Technical University of Athens, [cfrag@cs.ntua.gr](mailto:cfrag@cs.ntua.gr)

<sup>§</sup>Department of Informatics and Telecommunications, National University of Athens, [emarkou@di.uoa.gr](mailto:emarkou@di.uoa.gr)

ing a minimum number of guards placed on vertices; extensions consider edges or points in the interior. These problems are APX-hard and  $O(\log n)$ -approximable [8, 3, 4]. A related problem about terrain guarding, is the MINIMUM FIXED HEIGHT VERTEX (POINT) GUARDS ON TERRAIN problem ( $\Theta(\log n)$ -approximable [6], [3], [4]). In [7] the case of guarding the walls (and not necessarily every interior point) is studied. In [2] the following problem has been introduced: suppose we have a number of valuable treasures in a polygon  $P$ ; what is the minimum number of mobile (edge) guards required to patrol  $P$  in such a way that each treasure is always visible from at least one guard? In [2] they show NP-hardness and give heuristics for this problem. In [1] weights are assigned to the treasures in the gallery. They study the case of placing one guard in the gallery in such a way that the sum of weights of the visible treasures is maximized. Recent (non-)approximability results for art gallery problems can be found in [8, 11, 12, 13, 4, 6].

## 2 Finest Visibility Subdivision

We recall from [5] and [9] some preliminary definitions: Let  $P$  be a simple polygon,  $a, b \in P$  two points inside  $P$  and  $L, M \subseteq P$  two sets of points inside  $P$ . We say that point  $a$  sees point  $b$ , i.e.  $a$  and  $b$  are mutually visible, if the segment connecting  $a$  and  $b$  lies inside the closed polygon  $P$ . We say that the point set  $L$  is visible from the point set  $M$  or that  $M$  oversees  $L$  if for every point  $a$  which belongs to  $L$ , there exists a point  $b$  that belongs to  $M$ , such that  $a$  sees  $b$ . Notice that if  $M$  oversees  $L$ , it is not necessary for  $L$  to oversee  $M$ . Finally,  $M$  watches  $L$  if there exists a point  $a$  that belongs to  $L$  and a point  $b$  that belongs to  $M$  such that  $a$  sees  $b$ . Notice that if  $M$  watches  $L$  then also  $L$  watches  $M$ .

Our method descriptizes the interior of any simple polygon with respect to visibility. In [5] we defined the notion of the *Finest Visibility Segmentation* of the boundary of  $P$ : Consider the visibility graph  $V_G(P)$  with vertex set  $V(P)$ , i.e. the vertex set of  $P$ , where two vertices share an edge iff they are visible in  $P$ . By extending the edges of  $V_G(P)$  inside  $P$  up to the boundary of  $P$  we obtain a set of points  $FVS$  of the boundary of  $P$ , that includes of course all vertices of  $P$ . An extended edge of  $V_G(P)$  generates at most two  $FVS$  points. Since there are  $O(n^2)$  edges in  $V_G(P)$ , there are  $O(n^2)$  points in any polygon's  $FVS$  set. We

call this construction the *Finest Visibility Segmentation* of the boundary of  $P$ . Any *open* segment  $(a, b)$ , (i.e.  $a$  and  $b$  are excluded), defined by consecutive *FVS* points, is called an *FVS* segment of  $P$ .

Here we extend the *FVS* construction considering also all the intersection points of all the extended visibility graph's edges inside  $P$ . There are  $O(n^4)$  regions created inside  $P$  which are called *FVS* regions of  $P$ . Due to the above construction, such an *FVS* region has the following property: none of the extended visibility edges can cross the region. We call this construction the *Finest Visibility Subdivision* of the interior of  $P$ .

The following two lemmas establish that a *FVS* region cannot be **only partly** visible from a vertex or an edge.

**Lemma 1** *For any vertex  $v$  of a simple polygon  $P$ , a *FVS* region is visible by  $v$  if and only if it is watched by  $v$ .*

**Proof.** Of course if a *FVS* region is visible by  $v$ , then it is watched by  $v$ . Suppose now that the region is watched by  $v$  but not overseen by  $v$ . In that case there must be another vertex  $v'$  which blocks the visibility of  $v$ . But then  $vv'$  is a visibility edge and the extension of  $vv'$  crosses the *FVS* region which cannot hold since it contradicts the definition of a *FVS* region.  $\square$

**Lemma 2** *For any edge  $e$  of  $P$ , a *FVS* region is visible by  $e$  if and only if it is watched by  $e$ .*

**Proof. (Sketch)** Of course if a *FVS* region is visible by  $e$ , then it is watched by  $e$ . Suppose now that the region is watched by  $e = (v_i, v_j)$  but not overseen by  $e$ . This means that there is a point  $p$  inside the *FVS* region which sees a point  $d \in e$ . Sweeping the line that passes through  $p$  and  $d$  around  $d$  we meet a vertex  $v_m$  of  $P$  before the sweep line stops crossing the *FVS* region (since the region is not overseen). We start now sweeping the line that passes through  $d$  and  $v_m$  around  $v_m$  until the line stops crossing the *FVS* region. Notice that if another vertex  $v_n$  is hit by the sweep line then there must be a visibility edge (e.g.  $v_m v_n$ ) whose extension crosses the *FVS* region. But the latter cannot hold since by definition an *FVS* region is not crossed by any visibility edges.  $\square$

The above lemmas demonstrate the following: The interior of  $P$  can be effectively discretized in terms of visibility to  $O(n^4)$  *FVS* regions. Any vertex (edge) of  $P$  oversees a *FVS* region if and only if it watches the *FVS* region. In order to find the set of all overseen *FVS* regions from a polygon vertex  $v$ , namely the *FVS*( $v$ ) set, (using lemma 1) it suffices to select a point  $p$  inside every *FVS* region and connect it to vertex  $v$ . If this segment is everywhere inside the polygon  $P$ , then the region containing  $p$  is overseen from  $v$ .

For the case of the *FVS*( $e$ ) set, that is the set of all overseen *FVS* segments from a polygon edge  $e$ , (using lemma 2) it suffices to select a point  $p$  inside every *FVS* region and then sweeping a line around  $p$ . The *FVS* region containing  $p$  is overseen by an edge touched by that line.

### 3 The MAXIMUM AREA VERTEX (EDGE) GUARDS problem

Let  $A(r)$  be the area of region  $r$ .

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#### Algorithm 1 Maximum Area Vertex Guards

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compute the FVS regions
for all  $v \in V(P)$  do
  compute FVS( $v$ )
end for
 $SOL \leftarrow \emptyset$ 
for  $i = 1$  to  $k$  do
  select  $v \in V$  that maximizes  $A(SOL \cup FVS(v))$ 
   $SOL \leftarrow SOL \cup FVS(v)$ 
end for
return  $A(SOL)$ 

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Consider  $P$  and integers  $k, A > 0$ . It is NP-hard to decide whether we can place at most  $k$  guards on vertices of  $P$  so that the total area overseen by the guards is at least  $A$ . To see why, consider the decision MINIMUM VERTEX GUARDS problem, which asks whether the interior is overseen by at most  $k$  guards. The reduction to MAXIMUM AREA VERTEX GUARDS is straightforward.

In [10] they prove that to maximize the guarded boundary of a polygon with or without holes using guards placed on vertices or edges, is APX-hard. They present a gap preserving reduction from MAX-5-OCCURENCE-3-SAT to MAXIMUM VALUE VERTEX GUARD problem. In the construction part, methods from [8] for NP-hardness and [3] for APX-hardness are combined. If we change the construction part of the reduction so that to make sure that the area touched by the (so called) "cheap edges" is small enough then the following theorem holds:

**Theorem 3** *The MAXIMUM AREA VERTEX (EDGE) GUARDS problem is APX-hard.*

Alg. 1 is an approximation algorithm for the MAXIMUM AREA VERTEX GUARDS problem. It starts by calculating the *FVS* regions and then for every  $v \in V(P)$  the set *FVS*( $v$ ). During each iteration of the algorithm, for any vertex  $v$  that hasn't been assigned a guard yet, the set  $SOL \cup FVS(v)$  (of the overseen regions) is found and its area is calculated. The vertex that maximizes the total area of the (not previously) overseen regions is then chosen, causing a maximum possible increase of the solution. Then the

algorithm updates the set  $SOL$  by adding the new  $FVS$  regions.

In order to prove that algorithm 1 approximates MAXIMUM AREA VERTEX GUARDS by a constant approximation factor, we work as follows:

Let  $OPT$  denote the collection of the set of regions in an optimal solution and  $SOL$  denote the collection returned by the algorithm. These collections have  $A(OPT)$  and  $A(SOL)$  values respectively. Suppose that the algorithm places a guard at vertex  $v_i$  at iteration  $i$ , and a set of new regions  $P_i$ . Therefore the added total value of regions at iteration  $i$  is  $A(P_i)$ .

Consider the ordered sequence of vertices (as they have been selected by the algorithm) and let  $v_l$  be the first vertex in the sequence where a guard has been placed by the algorithm but not in the optimal solution. It holds:

$$A(P_i) = A(\cup_{m=1}^i P_m) - A(\cup_{m=1}^{i-1} P_m)$$

**Lemma 4** After  $l$  iterations of algorithm 1, we have, for  $l = 1, 2, \dots, k$ ,

$$A(\cup_{i=1}^l P_i) - A(\cup_{i=1}^{l-1} P_i) \geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k}.$$

**Proof.** Consider vertices where guards have been placed in the optimal solution but no guard has been placed there by the algorithm. By the pigeonhole principle, there is at least one such vertex  $v_m$  so that the following holds:

$$\begin{aligned} A(P'_m) &\geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P'_i)}{k} \Rightarrow \\ \Rightarrow A(P'_m) &\geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k}. \end{aligned}$$

Notice that  $A(P_l) \geq A(P_m) \geq A(P'_m)$  and  $A(\cup_{i=1}^l P_i) - A(\cup_{i=1}^{l-1} P_i) = A(P_l)$ , therefore:

$$A(\cup_{i=1}^l P_i) - A(\cup_{i=1}^{l-1} P_i) \geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k}$$

□

**Lemma 5** After  $l$  iterations of algorithm 1 it holds:

$$A(\cup_{i=1}^l P_i) \geq (1 - (1 - \frac{1}{k})^l)A(OPT), \quad l = 1, \dots, k$$

**Proof.** We are going to prove this by induction on  $l$ . During the first step of the algorithm the set with value  $A(P_1)$  is chosen. It holds:

$$A(P_1) \geq A(P'_1)$$

$A(P'_1)$  is the maximum possible value that  $OPT$  achieves, so from the pigeonhole principle:

$$A(P'_1) \geq \frac{A(OPT)}{k} \rightarrow A(P_1) \geq \frac{A(OPT)}{k}$$

Suppose that the relation holds for  $i = l - 1$ :

$$A(\cup_{i=1}^{l-1} P_i) \geq (1 - (1 - \frac{1}{k})^{l-1})A(OPT).$$

Since  $A(\cup_{i=1}^l P_i) = A(\cup_{i=1}^{l-1} P_i) + (A(\cup_{i=1}^l P_i) - A(\cup_{i=1}^{l-1} P_i))$ , using lem. 4 we have:

$$A(\cup_{i=1}^l P_i) \geq A(\cup_{i=1}^{l-1} P_i) + \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k} \rightarrow$$

$$A(\cup_{i=1}^l P_i) \geq A(\cup_{i=1}^{l-1} P_i)(1 - \frac{1}{k}) + \frac{A(OPT)}{k}$$

From the inductive hypothesis:

$$A(\cup_{i=1}^l P_i) \geq (1 - (1 - \frac{1}{k})^{l-1})A(OPT)(1 - \frac{1}{k}) + \frac{A(OPT)}{k}$$

$$\rightarrow A(\cup_{i=1}^l P_i) \geq (1 - (1 - \frac{1}{k})^l)A(OPT)$$

□

**Theorem 6** Algorithm 1 runs in polynomial time and achieves an approximation ratio of  $\frac{1}{1 - \frac{1}{e}} \approx 1.58$  for the MAXIMUM AREA VERTEX GUARDS problem.

**Proof.** Using lemma 5, we set  $l = k$  and get:

$$A(\cup_{i=1}^k P_i) \geq (1 - (1 - \frac{1}{k})^k)A(OPT)$$

It holds:

$$\lim_{k \rightarrow \infty} (1 - (1 - \frac{1}{k})^k) = 1 - \frac{1}{e}$$

As  $(1 - (1 - \frac{1}{k})^k)$  continuously gets smaller, we have:

$$1 - (1 - \frac{1}{k})^k \geq 1 - \frac{1}{e}$$

So:

$$A(SOL) > (1 - \frac{1}{e})W(OPT)$$

That is the algorithm approximates the MAXIMUM AREA VERTEX GUARDS problem with a  $\frac{1}{1 - \frac{1}{e}} \rightarrow 1.58$  ratio. □

The algorithm's complexity is  $O(n^4)$  because this is the size of  $FVS$ . A different approach, based on a triangulation and avoiding the  $FVS$ , yields  $O(n^2k^2)$  for the same algorithm.

Similarly as before the MAXIMUM AREA EDGE GUARDS problem is APX-hard. A similar algorithm to algorithm 1 approximates the MAXIMUM AREA EDGE GUARDS problem. The only difference from algorithm 1 is that we need to calculate the  $FVS(e)$  set using the technique described in section 2.

All algorithms apply when the polygons have holes.

#### 4 The MAXIMUM TREASURES VALUE VERTEX (EDGE) GUARDS problem

Consider the following problem: There exists a simple polygon  $P$  which encloses a number of simple subpolygons each of which has a non-negative value assigned. The MAXIMUM TREASURES VALUE VERTEX (EDGE) GUARDS problem's goal is to place vertex or edge guards in a way which maximizes the total value of the overseen or watched subpolygons. To show the usefulness of *FVS*, we prove the following: Given is  $P$ , which encloses a number of simple subpolygons, each of which has a value assigned. Given also are two integers  $k, M > 0$ . It is NP-hard to decide whether we can place at most  $k$  vertex or edge guards so that the total value of the overseen or watched subpolygons is at least  $M$ . The proof goes as follows: The decision version of MINIMUM VERTEX GUARDS problem for  $P$  reduces to the corresponding decision version of the MAXIMUM TREASURES VALUE VERTEX (EDGE) GUARDS problem for the same polygon: Construct all *FVS* regions of  $P$  and assign value 1 to each region; take as  $M$  the total number of *FVS* regions. Now the reduction is straightforward: The polygon's interior is overseen by at most  $k$  guards if and only if the total value of the overseen *FVS* regions is at least  $M$ .

A stronger negative result is the following:

**Theorem 7** MAXIMUM TREASURES VALUE VERTEX (EDGE) GUARDS is APX-hard.

For this proof, we can change the construction part of the reduction presented in [10] by adding small enough subpolygons touching the "cheap edges". Then, we assign a suitable small value to each subpolygon.

Alg. 1, with the appropriate modifications, approximates also the MAXIMUM TREASURES VALUE VERTEX (EDGE) GUARDS problem with the same ratio as in theorem 6. In fact, the computation of the *FVS* regions is not required for the case of vertex watching guards, since we can easily compute the subpolygons that are watched by a vertex guard. However, for the case of edge guards, a subpolygon  $p_i$  is watched by an edge  $e$  if and only if there is a *FVS* region watched by  $e$  that touches  $p_i$ . The total value of subpolygons in  $A(SOL \cup FVS(v))$  is also required.

Notice that all the algorithms can be applied even when the polygons have holes.

#### 5 Open problems

Interesting problems are the following: (a) How to place guards and given subpolygons in the polygon so that a maximum value is guarded (i.e. this time we need to place also the subpolygons). (b) How to place guards in the interior of  $P$  for all the above problems.

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