On Realistic Terrains*

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Abstract

We study worst-case complexities of the visibility map, the shortest path map, and the Voronoi diagram on terrains under realistic assumptions on edge length ratios and the angles of the triangles. We show that their complexities are considerably lower on realistic terrains than in the general case.

1 Introduction

One of the main objectives of computational geometry is to uncover the computational complexity of geometric problems. It provides a theory that explains how efficiently geometric problems can be solved that arise in applications. However, in many cases a discrepancy exists between the provable worst-case computational complexity of an algorithm and the actual running time behavior of that algorithm on inputs that arise in applications. This has led to the study of *fatness* and *realistic input models*.

Among the first applications of realistic input models in computational geometry, Alt *et al.* [1] consider motion planning for a rectangular robot. The efficiency depends on the aspect ratio of this rectangle. Matoušek *et al.* [8] show that if all triangles of a set of n triangles have their angles bounded away from zero (at least α, for some constant $\alpha > 0$), then the union of these triangles has $O(n)$ holes rather than $O(n^2)$ for the general case, and the boundary complexity of the union is $O(n \log \log n)$. Such triangles are called *fat*. Since then, various definitions of fatness [2, 5, 12, 14] have been proposed. Other realistic models—such as low density [13], unclutteredness [4], and simple-cover complexity [10]—consider the spatial distribution of objects or their features. An overview of reduced combinatorial complexities and improved algorithmic efficiencies for inputs satisfying these models is given in [4] along with a model hierarchy.

Realistic assumptions have not yet been studied for polyhedral terrains. However, several geometric structures on terrains have complexities much higher than typical in applications. For example, the visibility map of a polyhedral terrain of n triangles has complexity $\Theta(n^2)$ in the worst case, a shortest path has $\Theta(n)$ complexity, and even a bisector of two points on a terrain can have quadratic size. Hence, the discrepancy between theoretical complexity bounds and typical complexity bounds exists on terrains as well. In this paper, we analyze this discrepancy by studying realistic assumptions on terrains.

For visibility maps, we give three assumptions whose combination provides an $O(n\sqrt{n})$ bound on the visibility map of a terrain when viewed from infinity, and an $O(n\sqrt{n} \log \log n)$ bound for perspective views. Dropping any of the three assumptions immediately makes an $\Omega(n^2)$ lower bound construction possible. With our three assumptions, we provide a lower bound construction of size $\Omega(n\sqrt{n})$, matching the upper bound for views from infinity. It is interesting to note that the assumptions all refer to the xy-projection of the terrain.

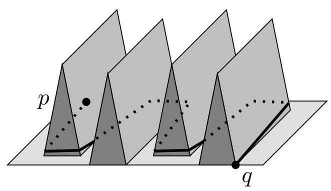

Figure 1: A terrain where in the projection, all its vertices lie on a regular grid and the shortest path between p and q crosses $\Omega(n)$ triangles.

Next, we consider distance structures on terrains. Using only the three assumptions for visibility, we can still have shortest paths that have linear complexity; see Figure 1. Therefore, we introduce a fourth assumption that relates to the steepness of the terrain, and show that any shortest path between two points passes through only $\Theta(\sqrt{n})$ triangles. For a bisector between two points, we show that the same set of four assumptions gives an $O(n\sqrt{n})$ complexity bound rather than quadratic. We give an $\Omega(n)$ size lower bound. The shortest path map for a source point s is the subdivision of the terrain into regions where the combinatorial structure of shortest paths from s is the same. In general it has complexity $\Theta(n^2)$, but we show that under our assumptions it is $\Theta(n\sqrt{n})$. Finally, we study Voronoi diagrams on terrains, which in general also have quadratic complexity, even for only two sites. Our assumptions allow us to prove an upper bound of $O(n\sqrt{n})$, and we give a lower bound of $\Omega(n+m\sqrt{n})$ in case there are m sites on the terrain.

Algorithmically, our results imply faster compu-

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tation of visibility maps on realistic terrains. The output-sensitive construction of the visibility map of a terrain by Katz *et al.* [7] implies that for realistic terrains, it can be computed in $O(n\sqrt{n}\log n)$ time for views from infinity, and in $O(n\sqrt{n}\log n \log\log n)$ time for perspective views.

Shortest paths on general terrains can be computed in $O(n\log^2 n)$ time [6]. This improved earlier algorithms [3, 9]. The description of Kapoor [6] does not provide sufficient detail to analyze whether a slightly faster algorithm can be obtained for realistic terrains. For shortest path maps and Voronoi diagrams, the algorithm of Mitchell *et al.* [9] leads to $O(n\sqrt{n}\log n)$ time bounds for the construction on realistic terrains.

The proofs and lower bound constructions that we omit in this extended abstract can be found in the full version of this paper [11].

2 Realistic model

Let $\mathcal{T}$ be a polyhedral terrain, comprising a set T of n triangles, a set E of n_e edges, and a set V of n_v vertices, where n_e and n_v are $O(n)$. We assume $\mathcal{T}$ satisfies the following three properties:

1. the minimum angle of any triangle in T in the projection to the xy-plane is at least α,
2. the boundary of the projection of $\mathcal{T}$ onto the xy-plane is a rectangle with side lengths 1 and c,
3. the longest xy-projection over all edges in E is at most d times as long as the shortest one.

The values α, c, and d in the above assumptions are all positive constants. A projected triangle that satisfies the first assumption is *fat* [8]. We call a polyhedral terrain $\mathcal{T}$ a *realistic terrain* if $\mathcal{T}$ satisfies assumptions 1, 2, and 3. Realistic terrains have certain properties, stated in the lemmas and corollaries below, which we use in the next section to prove upper bounds on the worst-case complexities of visibility structures.

Lemma 1 *Every edge of a realistic terrain has length* $\Theta\left(\frac{1}{\sqrt{n}}\right)$ *in the projection.*

Corollary 2 *Every triangle in a realistic terrain has area* $\Theta\left(\frac{1}{n}\right)$ *in the projection.*

Lemma 3 *Two vertices v and w in a realistic terrain have distance* $\Omega\left(\frac{1}{\sqrt{n}}\right)$ *in the projection.*

Corollary 4 *Two edges e and e' in a realistic terrain that have no common endpoint have distance* $\Omega\left(\frac{1}{\sqrt{n}}\right)$ *in the projection.*

The terrain in Figure 1 satisfies the three assumptions above, but a shortest path between two points on the terrain still passes through $\Theta(n)$ triangles in the worst case. In Section 4, we discuss distance structures on terrains, and to bound their complexity, we introduce an additional assumption:

4. the dihedral angle of the supporting plane of any triangle in T with the xy-plane is at most β, where $\beta < \frac{\pi}{2}$ is some constant.

Assumption 4 implies that the maximum slope of a line segment on any triangle of $\mathcal{T}$ is $\tan\beta = O(1)$. In Section 4, we call a terrain *realistic* if it satisfies all four assumptions.

All complexity bounds for the visibility map of a realistic terrain in Section 3 are achieved with only the first three assumptions. In particular, we do not need a bound on the dihedral angles to prove the upper bounds, and the lower bound constructions are all possible with bounded dihedral angle. In the full version of this paper [11], we show that the first three assumptions are necessary to obtain a subquadratic upper bound on the complexity of the visibility map.

3 Complexity of the visibility map

Let p be the viewpoint of the visibility map $\mathrm{VM}(p,\mathcal{T})$ that we consider. We bound the complexity of the visibility map by giving an upper bound on $\#I_t$, the number of terrain triangles with which a given triangle t can *interact*, i.e., with which it can create features of $\mathrm{VM}(p,\mathcal{T})$. Recall that a vertex of the visibility map of p directly corresponds to a line through p that is a common tangent of two terrain edges. Since any two triangles create $\Theta(1)$ vertices of $\mathrm{VM}(p,\mathcal{T})$ in the worst case, we get the following expression:

$$\text{Complexity of } \mathrm{VM}(p,\mathcal{T}) = O\left(\sum_{t\in T} \#I_t\right).$$

We call the locus of the triangles with which a triangle t interacts in the visibility map of p the *influence region* of t, and we denote it by $\mathcal{R}_t$. Using this definition, we can bound $\#I_t$ from above by the number of triangles in T whose projection intersects $\mathcal{R}_t$. We distinguish two cases based on the location of the viewpoint: (a) p is located infinitely far away from $\mathcal{T}$ (parallel projection), and (b) p lies on or above $\mathcal{T}$ (perspective projection).

In case (a), for any triangle t, all triangles that intersect the influence region $\mathcal{R}_t$ are contained in a slab bounded by two parallel lines, whose width is three times the length of the longest edge in E; see Figure 2(a). We give bounds on the complexity of $\mathrm{VM}(p,\mathcal{T})$ in this case in Section 3.1. In case (b), the influence region of a triangle t is a truncated wedge; see Figure 2(b). In Section 3.2, we bound the complexity of $\mathrm{VM}(p,\mathcal{T})$ in this case.

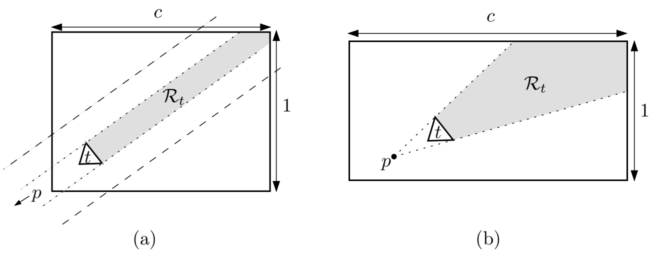

Figure 2: The influence region for a triangle t and (a) a viewpoint at infinity or (b) a viewpoint on or above the terrain.

3.1 Viewpoint at infinity

Under the assumptions of Section 2, we can place $\Theta(\sqrt{n})$ triangles in a rectangle of constant length and of width $\Theta(\frac{1}{\sqrt{n}})$. By placing $\Omega(\sqrt{n})$ triangles at one end of this rectangle, each of which interacts with $\Omega(\sqrt{n})$ triangles at the other end, we can get a parallel projection with complexity $\Omega(n)$ for this rectangle.

Since the projection of $\mathcal{T}$ is a rectangle with side lengths 1 and c, we can replicate this construction $\Omega(\sqrt{n})$ times, resulting in a visibility map of complexity $\Omega(n\sqrt{n})$ in total; see Figure 3(a).

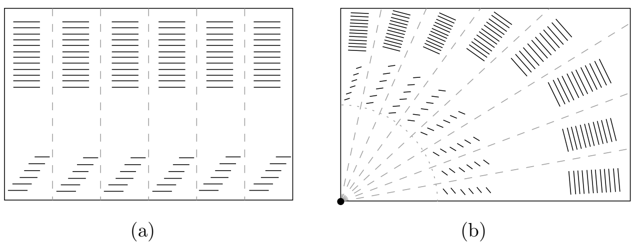

Figure 3: The visibility map of (a) a viewpoint at infinity, or (b) a viewpoint on or above the terrain, can have $\Omega(n\sqrt{n})$ vertices.

Lemma 5 *The visibility map of a realistic terrain for a viewpoint at infinity can have $\Omega(n\sqrt{n})$ vertices.*

We now show that this bound is tight. Let l_1 and l_2 be two parallel lines in the xy-plane, both intersecting the projection of $\mathcal{T}$, at distance three times the length of the longest edge in E. Let L be the slab that is formed by l_1 and l_2. For every triangle t of T and every viewpoint at infinity, there exists such a slab L_t that completely contains all the triangles that intersect $\mathcal{R}_t$. By Lemma 1, for any triangle t from T, the area of L_t is $O(1/\sqrt{n})$, and thus, by Corollary 2, the number of triangles in L_t is $O(\sqrt{n})$. Now, by the discussion at the beginning of Section 3 and Lemma 5, we have the following theorem.

Theorem 6 *Let $\mathcal{T}$ be a realistic terrain with n triangles, and let p be a viewpoint at infinity. Then $\mathrm{VM}(p, \mathcal{T})$ has complexity $\Theta(n\sqrt{n})$ in the worst case.*

3.2 Viewpoint on or above the terrain

We can create a subconstruction with $\Omega(\sqrt{n})$ triangles that are located in a wedge of area $\Omega(1/\sqrt{n})$, instead of in a rectangle of the same area, as is the case in Figure 3(a). If we place the viewpoint at the apex of the wedge, then this subconstruction produces a visibility map with $\Omega(n)$ vertices. We can replicate this construction $\Omega(\sqrt{n})$ times in a rectangle of $\Theta(1)$ area; Figure 3(b) displays the construction schematically.

Lemma 7 *The visibility map of a realistic terrain for a viewpoint on or above the terrain can have $\Omega(n\sqrt{n})$ vertices.*

To obtain an almost matching upper bound, we subdivide the projection of $\mathcal{T}$ into annuli of increasing size and increasing distance from p and bound the number of visibility map vertices for every region separately; this captures the intuition that triangles far away from the viewpoint contribute less to the visibility map than triangles close to the viewpoint. In this way, we can prove the following lemma.

Lemma 8 *The visibility map of a point on or above a realistic terrain has complexity $O(n\sqrt{n}\log\log n)$.*

Summarizing the results in this section gives us the following theorem.

Theorem 9 *Let $\mathcal{T}$ be a realistic terrain with n triangles, and let p be a point located on or above $\mathcal{T}$. Then $\mathrm{VM}(p, \mathcal{T})$ has worst-case complexity $\Omega(n\sqrt{n})$ and $O(n\sqrt{n}\log\log n)$.*

4 Complexity of the shortest path map and the Voronoi diagram

Shortest paths on terrains were studied extensively in [3, 6, 9]. It is easy to see that on any realistic terrain $\mathcal{T}$, two points exist whose shortest path passes through $\Omega(\sqrt{n})$ triangles. We show a matching upper bound. First, we can show that the shortest path between two points on a realistic terrain has constant length, because it cannot be more than a constant times as long as its xy-projection. By the results of Section 2, this leads to the following lemma.

Lemma 10 *Let $\mathcal{T}$ be a realistic terrain with n triangles. The shortest path over $\mathcal{T}$ between two points p and q on $\mathcal{T}$ passes through $\Theta(\sqrt{n})$ triangles in the worst case.*

The bisector $B(p, q)$ of a point p and a point q on $\mathcal{T}$ is the set of points on $\mathcal{T}$ with equal distance to p and q. It is a simple curve (open or closed) that consists of line segments and hyperbolic arcs [9]. The worst-case complexity of a bisector is $\Theta(n^2)$ on general terrains. The bisector $B(p, q)$ has $O(n)$ *breakpoints*, which are the points that have two shortest paths to p or two shortest paths to q. In [11], we show that on realistic terrains, a bisector intersects $\Theta(\sqrt{n})$ triangles between two breakpoints in the worst case.

Lemma 11 *For two points p and q on a realistic terrain $\mathcal{T}$, the bisector $B(p,q)$ has complexity $O(n\sqrt{n})$.*

The *shortest path map* of a source point s is the subdivision of $\mathcal{T}$ into cells such that the vertex and edge sequence of the shortest path to any point in that cell from the source is the same. In general terrains, this structure has worst-case complexity $\Theta(n^2)$ [9]. A global analysis of where the boundaries of the shortest path map cells originate from, yields an $O(n\sqrt{n})$ size bound for realistic terrains.

Theorem 12 *The shortest path map of a realistic terrain has complexity $\Theta(n\sqrt{n})$ in the worst case.*

Although we could not use the combinatorial analysis of Mitchell *et al.* [9] to obtain better complexity bounds, it is easy to verify that the total number of points for which their algorithm computes the additively weighted Voronoi diagram is $O(n\sqrt{n})$ on a realistic terrain, and therefore the shortest path map for a point can be computed in $O(n\sqrt{n}\log n)$ time.

We are also interested in the maximum complexity of the Voronoi diagram of a set S of m sites on a realistic terrain $\mathcal{T}$, where distances are shortest path distances on $\mathcal{T}$. The Voronoi cell of a site $s_i \in S$ on $\mathcal{T}$ is connected, but not necessarily simply-connected. As a consequence, only $O(m)$ bisectors appear in the Voronoi diagram of S on $\mathcal{T}$, and there are $O(m)$ Voronoi vertices. This immediately leads to an $O(mn\sqrt{n})$ bound on the complexity, but in [11], we show that it is $O(n\sqrt{n})$.

Since the bisector of two sites can have $\Omega(n)$ breakpoints, this is a trivial lower bound. Alternatively, we can place m sites on a terrain such that projected, all $m-1$ bisectors are lines parallel to the y-axis. Each bisector intersects $\Omega(\sqrt{n})$ triangles between the boundaries with the terrain, which gives a Voronoi diagram of complexity $\Omega(m\sqrt{n})$.

Theorem 13 *The Voronoi diagram of a set of m sites on a realistic terrain has complexity $\Omega(n+m\sqrt{n})$ and $O(n\sqrt{n})$ in the worst case.*

5 Concluding Remarks

This paper studied realistic input models for polyhedral terrains, a topic that has not been considered so far. We have made three input assumptions that together are necessary and sufficient to show a subquadratic upper bound on the complexity of the visibility map. For paths, bisectors, shortest path maps, and Voronoi diagrams on terrains, we used a fourth input assumption and proved upper and lower bounds on their complexities. Our research helps to explain the discrepancy between the worst-case performance of algorithms on polyhedral terrains and their efficiency in practice.

The upper and lower bounds for visibility maps, shortest paths, and the shortest path map are tight or nearly tight, but there is a considerable gap for bisectors and Voronoi diagrams. This is the most important open problem that arises from this paper.

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