An homotopy theorem for arrangements of double pseudolines

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Abstract

We define a *double pseudoline* as a simple closed curve in the open Möbius band homotopic to the double of its core circle, and we define an *arrangement of double pseudolines* as a collection of double pseudolines such that every pair crosses in 4 points – the crossings being transversal – and induces a cell decomposition of the Möbius band whose 2-dimensional cells are 2-balls, except the unbounded cell which is a 2-ball minus a point. Dual arrangements of boundaries of collection of pairwise disjoint 2-dimensional closed bounded planar convex sets are examples of arrangements of double pseudolines. We show that every pair of simple arrangements of double pseudolines is connected by a sequence of triangle-switches and that every simple arrangement of double pseudolines has a *representation* by a configuration of pairwise disjoint disks in the plane with pseudoline double tangents. This shows in particular that any double-permutation sequence of J.E. Goodman and R. Pollack (SoCG'05 page 159, [2]) has a representation by a configuration of pairwise disjoint disks in the plane with pseudoline double tangents. We also present some enumeration results for our arrangements, and a property of their subarrangements.

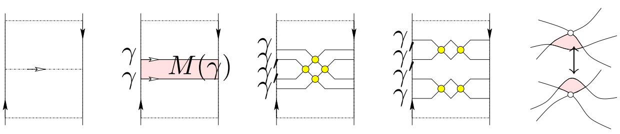

Figure 1: The Möbius band and its core circle, a (monotone) double pseudoline γ and the Möbius band $M(\gamma)$ bounded by γ, an arrangement of two double pseudolines γ and γ', a collection of two double pseudolines with 4 crossing points but which is not an arrangement because the cell intersection of the associated Möbius bands is not a 2-ball, and a triangle-switch.

1. Arrangements of double pseudolines in the Möbius band. Let $\mathbb{M}$ be the open Möbius band, say quotient of $\mathbb{R}^2$ under the map $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ that assigns to the pair (θ, u) the pair $(\theta + \pi, -u)$, and let $c : [0, 1] \to \mathbb{M}$, $c(t) = (\pi t, 0)$, be its core circle. A *pseudoline* in $\mathbb{M}$ is a simple closed path in $\mathbb{M}$ homotopic to its core circle and a *double pseudoline* in $\mathbb{M}$ is a simple closed path in $\mathbb{M}$ homotopic to the double cc of its core circle. Boundary curves of tubular neighborhoods of pseudolines are examples of double pseudolines. We define an *arrangement of double pseudolines* in $\mathbb{M}$ as a finite collection Γ of double pseudolines in $\mathbb{M}$ such that every pair of elements of Γ crosses in 4 points – the crossings being transversal – and induces a cell decomposition of $\mathbb{M}$ whose 2-dimensional cells are 2-balls, except the unbounded cell which is a 2-ball minus a point, and we define the *chirotope* χ_Γ of Γ as the map that assigns to each $\gamma \in \Gamma$ and to each ordered triple u, v, w of vertices lying on γ of the cell decomposition X_Γ of $\mathbb{M}$ induced by the elements of Γ the value $+1$ if walking along the curve γ we encounter the vertices in cyclic order $uvwuvw\cdots$; -1 otherwise. An arrangement of double pseudolines is called *simple* if exactly two double pseudolines meet at every vertex of the induced cell decomposition of $\mathbb{M}$. A simple arrangement of double pseudolines is called *thin* if there is no vertex of the induced cell decomposition lying in the interiors of the closed Möbius bands bounded by the double pseudolines. Thin arrangements are obtained from simple (finite) arrangements of pseudolines by replacing the pseudolines by the boundary curves of suitable tubular neighborhoods. Collections of dual curves[1] of boundaries of pairwise disjoint 2-dimensional closed bounded planar convex sets are examples of arrangements of double pseudolines; these arrangements are simple and thin under the additional assumption that there is no line transversal to three convex sets; these arrangements are also *monotone* with respect to the core circle in the sense every meridian $\theta = c$ of the Möbius band crosses every double pseudoline exactly twice. Finally we observe that, as in the case of arrangements of pseudolines, the set of arrangements of double pseudolines is stable by triangle-switch. The main result of the paper is the following.

Theorem 1 *Let Γ be a simple arrangement in $\mathbb{M}$ of double pseudolines, X the induced cell decomposition of $\mathbb{M}$, and $\gamma \in \Gamma$. Assume that there is a vertex of X lying in the interior of the Möbius band $M(\gamma)$ bounded by γ. Then there is a triangular face of X*

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[1] The dual of a planar smooth curve is the curve in the space of undirected lines of the plane of the tangent lines to the curve. We identify the space of undirected lines of the plane with the Möbius band $\mathbb{M}$ via the map that assigns to the pair (θ, u) the line with equation $u - x\sin\theta + y\cos\theta = 0$.

included in $M(\gamma)$ with a side supported by γ. □

Therefore triangle-switches are always possible for non thin arrangements of double pseudolines and, consequently, every arrangement of double pseudolines is homotopic to a “core” thin one; furthermore since thin arrangements are arrangements of boundary curves of tubular neighborhoods of simple arrangements of pseudolines and since Ringel’s homotopy theorem for arrangements of pseudolines [1, page 267] [5, 6] asserts that the set of simple arrangements of pseudolines is connected by triangle-switches we get a similar result for the set of arrangement of double pseudolines on a fixed number of double pseudolines. We summarize.

Corollary 2 *Let Γ be a simple arrangement of double pseudolines. Then there exists an homotopy Γ_t, $t \in [0,1]$, of $\Gamma = \Gamma_0$ onto a thin arrangement of double pseudolines Γ_1 such that $M(\gamma_t) \supseteq M(\gamma_{t'})$ for all $t \leq t'$. (Simplicity is of course not maintained during the homotopy.)* □

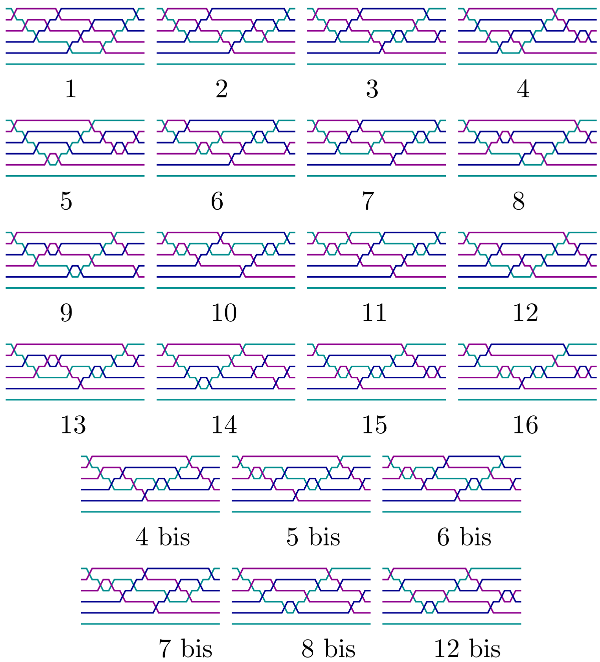
Corollary 3 *Let Γ and Γ' be two arrangements of n double pseudolines. Then Γ and Γ' are homotopic.* □

2. Combinatorial equivalence and enumeration results. For Γ an arrangement of two double pseudolines γ and γ' we set $v(\gamma, \gamma')$ to be the linear sequence of vertices of X_Γ encountered when walking along the curve γ starting from the source of the 2-cell intersection of the Möbius bands $M(\gamma)$ and $M(\gamma')$ associated with γ and γ'. Let Γ and Γ' be two arrangements of n double pseudolines and let $\varphi : \Gamma \to \Gamma'$ be a bijection. We denote by $\tilde{\varphi} : \Gamma \cup X_0 \to \Gamma' \cup X'_0$ the canonical extension of φ to the sets of vertices of X_Γ and $X_{\Gamma'}$ defined by the condition that $\tilde{\varphi}$ assigns to the sequence $v(\gamma_1, \gamma_2)$ the sequence $v(\varphi\gamma_1, \varphi\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$. We say that Γ and Γ' are *φ-equivalent* if $\chi_{\Gamma'} \circ \tilde{\varphi} = \chi_\Gamma$, that Γ and Γ' are *combinatorially equivalent* if Γ and Γ' are φ-equivalent for some bijection $\varphi : \Gamma \to \Gamma'$, and that Γ and Γ' are *combinatorially equivalent up to (global) reorientation* if Γ is equivalent to Γ' or to Γ'^{-1}. Finally we say that two ordered arrangements of n double pseudolines are *combinatorially equivalent* if they are φ-equivalent for the bijection φ induced by the orderings. We denote by a_n the number of classes of ordered simple arrangements of n double pseudolines, and by b_n et c_n the numbers of classes of simple arrangements of n double pseudolines under the combinatorial equivalence and combinatorial equivalence up to (global) reorientation.

Our main result provides an algorithm to enumerate the combinatorial equivalence classes of ordered arrangements and arrangements of n double pseudolines by a traversal of the triangle-switch graph of double pseudoline arrangements. We have implemented this algorithm. Preliminary results are reported in the following table that confirms the result reported in [8] concerning the value of c_4.

n	2	3	4
a_n	1	118	–
b_n	1	22	22620
c_n	1	16	11502

The numbers a_n, b_n et c_n are in $2^{\Theta(n^2)}$ since the cell decomposition X_Γ induced by an arrangement Γ of size n is constructable in time $O(n^2)$ using an incremental ramdomized algorithm [7]. Lower bounds were established (even for dual arrangements of convex sets) in [3, 4]. Representatives of the $22 = 16 + 6$ classes of combinatorially equivalent arrangements of three double pseudolines are depicted in the figure below. The 10 arrangements numbered 1,2,3,9,10,11,12,13,14,15 and 16 are combinatorially equivalent to their (global) reoriented versions. The reoriented versions of the arrangements numbered 4,5,6,7,8 and 12 are numbered 4 bis, 5 bis, 6 bis, 7 bis, 8 bis and 12 bis.



3. Representation by arrangements of disks. We use the following terminology. A *disk* is 2-dimensional bounded closed simply connected subset of the affine oriented plane. A *tangent* to a disk is a pseudoline that intersects the disk at a sole boundary point; in particular a disk is included in the closure of one of the two connected components of the complement of any of its tangents (half-plane for short). An *arrangement* of disks is a finite collection Δ of pairwise disjoint disks equipped with a map that assigns to each unordered pair o, o' of disks a set $L(o, o')$ of four double tangents to o and o' such that the whole set of pseudolines $L(\Delta)$ union of the $L(o,o')$'s is an affine arrangement of pseudolines and such that the intersection of a disk and a line is connected. We denote by $L(o)$ the set of pseudolines of $L(\Delta)$ that are tangents to o. An arrangement of disks is called *simple* if every tangent of the arrangement is tangent to exactly two disks. Wlog we will assume that a touching point between a double tangent and a disk lies on exactly one double tangent. The *chirotope* of an arrangement of disks Δ is the map χ_Δ that assigns to each disk o and to each ordered triple u, v, w of tangents to o the value +1 if walking counterclockwise around the boundary of the disk o we encounter the tangents in cyclic order $uvwuvw\cdots$; -1 otherwise.

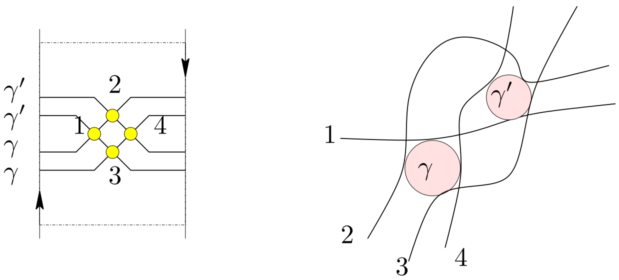


Figure 2: Local representation : the double pseudolines γ and γ' are represented by the disks labelled γ, γ' and the 4 vertices numbered 1,2,3, and 4 of the induced cell decomposition of $\mathbb{M}$ are represented by the pseudoline double tangents numbered 1,2,3 and 4.

Let now Γ be an arrangement of double pseudolines. An arrangement Δ of disks is called a *representation* of Γ if Δ and Γ have the same chirotope, i.e., there is a bijection $\varphi : \Gamma \to \Delta$ such that its extension $\varphi : X^0 \to L$ between the set X^0 of vertices of X_Γ and the set of double tangents L of Δ defined in Figure 2 carries the chirotope of Γ onto the chirotope of Δ, i.e., $\chi_\Delta \circ \varphi = \chi_\Gamma$.

Theorem 4 *Every simple arrangement of double pseudolines is representable by an arrangement of disks.*

Proof. (Sketch.) Since representable arrangements of double pseudolines exist it is sufficient, thanks to Theorem 1, to show that the property to have a representation is maintained during a triangle-switch operation. So let Γ be a simple arrangement of double pseudolines represented by an arrangement Δ of disks, i.e., there exists a bijection $\varphi : \Gamma \to \Delta$ such that $\chi_\Delta \circ \varphi = \chi_\Gamma$. Let σ a triangular face $\alpha\beta\gamma$ of X whose edges $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$ are supported by the double pseudolines U, V and W. Using a simple perturbation argument it is sufficient to prove that the (non simple) arrangement of double pseudolines Γ' obtained by collapsing the triangular face $\alpha\beta\gamma$ to a single point is representable by an arrangement of disks Ω'. The disks and double tangents corresponding to $U, V, W, \alpha, \beta, \gamma$ will be simply denoted $U^*, V^*, W^*, \alpha^*, \beta^*$ and γ^*. We orient the double tangent α^* from V^* towards U^* and β^* and γ^* from U^* towards W^*. In that case a simple analysis shows that the relative positions of the disks and the double tangents are as indicated in the top left part of Figure 3. (There are $2^3 = 8$ combinatorially different cases which correspond to the choice of the positions of the disks with respect to the double tangents.)

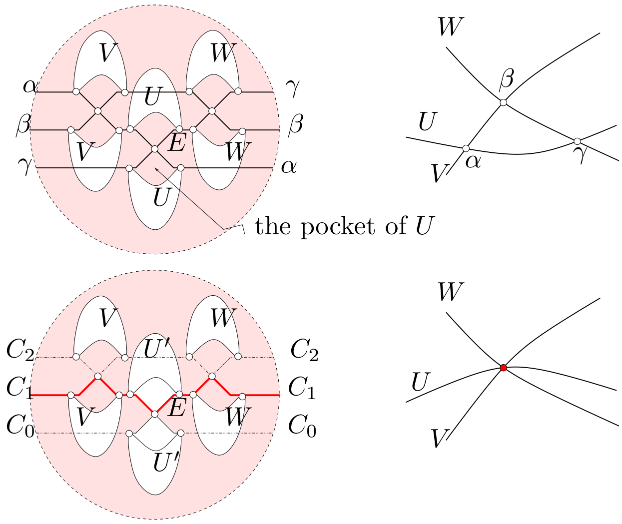


Figure 3: The three double pseudolines α^*, β^* and γ^* are replaced by the 1-level of their arrangement.

Let C_0, C_1 and C_2 be the 0-, 1- and 2-level of the arrangement of the pseudolines α^*, β^* and γ^*. We set $L' = L \setminus \{\alpha^*, \beta^*, \gamma^*\} \cup \{C_1\}$ and we define U' to be the union of the disk U^* and the "pocket" delimited by the contact points with U^* of the double tangents α^* and γ^* and their intersection point E and we define U'' to be a slight perturbation of U' so that its intersection with C_1 reduces to the vertex E (this pertubation has do be done only if U^* lies on the right sides of γ^* and α^*); similar constructions are done with the disks V^* and W^*. We set $\Omega' = \Omega \setminus \{U, V, W\} \cup \{U'', V'', W''\}$. One can check that L' is an arrangement of pseudolines and that the collection of disks Ω' equipped with the set L' is an arrangement of disks whose chirotope is combinatorially equivalent to the chirotope of Γ'. ☐

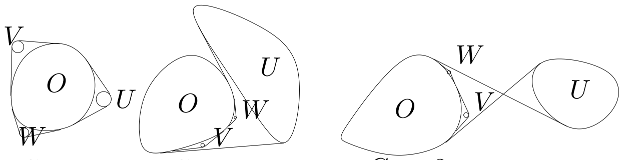
4. Wiring diagrams and double-permutation sequences. Since the triangle-switch operation can be implemented to preserve the monotonicity of the curves with respect to the core circle we see that every simple arrangement Γ of double pseudolines is combinatorially equivalent to a monotone one. Therefore one can not only speak of the source, $\mathrm{sour}(\sigma)$, and the sink, $\mathrm{sink}(\sigma)$, of every 1-cell or 2-cell $\sigma \in X_\Gamma$ but the transitive closure of the covering relations $\mathrm{sour}(\tilde\sigma) \prec \tilde\sigma \prec \mathrm{sink}(\tilde\sigma)$ defined on the lifts of the cells of X_Γ in the universal covering $p : \mathbb{R}^2 \to \mathbb{M}$ of $\mathbb{M}$ is a well-defined *partial order* on $\tilde{X}_\Gamma$. The following theorem provides a complete description of the set of monotone arrangements combinatorially equivalent to a given simple arrangement and gives an interpretation in terms of representation by arrangements of disks (answering positively a question set in [2, Remark 20] regarding the realizability of a double-permutation sequence by an arrangement of disks).

Theorem 5 *Let Γ be a monotone arrangement of n double pseudolines in $\mathbb{M}$ and assume that walking along the core circle we encounter the vertical projections v'_i of the vertices v_i of X_Γ in the circular order $v'_1 v'_2 \dots v'_{2n(n-1)}$. Then (1) $v_1 v_2 \dots v_{2n(n-1)}$ is the projection on $\mathbb{M}$ of a linear extension of the poset $(\tilde{X}^0_\Gamma, \prec)$ compatible with ι, i.e., if v precedes w then $\iota(v)$ precedes $\iota(w)$; (2) Γ is representable by an arrangement Δ of n disks such that the circular ordering of the corresponding double tangents with respect to the line at infinity corresponds to the circular ordering of the vertices.* □

5. On the chirotopes of subarrangements. By definition the chirotope of an arrangement of n disks depends only on its $\binom{n}{4}$ subarrangements of 4 disks. Since chirotopes of collections of points depends only by definition of the collection of chirotopes of subsets of three points it is natural toask if the same holds for arrangements of disks. The answer is yes. This result was conjectured[2] to be true for stretchable arrangements in [3, Theorem 3.] and checked using the exhaustive list of 11502 arrangements on 4 disks generated by triangle-switches starting from a stretchable arrangement by F. Torossian [8], ten years ago. We provide a direct proof below; this direct proof confirms partially the validity of our implementation.

Theorem 6 *The chirotope of an arrangement of n disks depends only on the $\binom{n}{3}$ chirotopes of its subarrangements of 3 disks.*

Proof. (Sketch.) Let O, U, V, W be 4 disks. We denote by χ the chirotope and χ^3 its restriction to triple of disks. We write $\mathrm{L}(X)$ for the set of double tangents to O and X, and we write (a, b) for the set of x such that $\chi_O(a, x, b)$. Let $u, u' \in \mathrm{L}(U)$ and let $x \in \mathrm{L}(U, V, W)$. Clearly one can decide if $\chi(u, x, u')$ using only χ^3. A pair (u, u') of bitangents is said to separate the pair (x, x') if u, u', x, x' appear in the cyclic order $uxu'x'$ around the disk O. Assume that a pair (u, u') of elements of $\mathrm{L}(U)$ separates a pair (x, x') of elements of $\mathrm{L}(V, W)$. In that case a triplet a, b, c of elements of $\mathrm{L}(V, W)$ lying in the interval (u, u') appears in the linear order abc in the interval (u, u') iff. x', a, b, c appear in the cyclic order $x'abc$ around the disk o. So it remains to examine the case where no pair of elements of $\mathrm{L}(U)$, $\mathrm{L}(V)$ and $\mathrm{L}(W)$ separates a pair of elements of $\mathrm{L}(V, W)$, $\mathrm{L}(W, U)$ and $\mathrm{L}(U, V)$. In that case $\chi(u, v, w)$ is independant of the choice of $u \in \mathrm{L}(U), v \in \mathrm{L}(V)$ and $w \in \mathrm{L}(W)$. We will simply write $\chi(U, V, W)$.



Case 1 Case 2 and 4 Case 3

Let 1,2,3 and 4 be the consecutive elements of $\mathrm{L}(U)$ on O where 1 is the right-left double tangent oriented from O toward U. We write $\mathrm{I}_1(U), \mathrm{I}_2(U), \mathrm{I}_3(U), \mathrm{I}_4(U)$ for the interval $(3, 4), (4, 1), (1, 2), (2, 3)$. It remains four cases to examine (cf. figure above):
case 1. $\mathrm{L}(V, W) = \mathrm{I}_4(U)$. In that case $\chi(U, V, W)$ iff U, V et W appear in this order when walking counterclockwise around the boundary of their convex hull.
case 2. $\mathrm{L}(V, W) = \mathrm{I}_1(U)$. In that case $\chi(U, V, W)$ iff W lies in the interior of the convex hull of U and V;
case 3. $\mathrm{L}(V, W) = \mathrm{I}_2(U)$. In that case $\chi(U, V, W)$ iff U, W, V appear in this in order when walking counterclockwise on the boundary of their convex hull.
case 4. $\mathrm{L}(V, W) = \mathrm{I}_3(U)$. similar to case 2 modulo a reorientation of the plane.

Furthermore in each of these four cases the chirotope is unique up to the permutation of V and W. □

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[2] Claimed without proof to be more honest!