Tight planar packings of two trees

Yoshiaki Oda[*] Katsuhiro Ota[†]

Abstract

When we consider an embedding of graphs into a plane, it would be nice if it does not intersect internally since the embedding simply shows us the structure of graphs. It is easy to embed a tree into a plane with non-self-intersections. If we embed two or more trees into a plane with non-self-intersections, what occurs? In this paper, we consider embeddings of two trees into a plane with using same vertices, sharing no edges and no-intersections. We prove that a non-star tree and a non-star caterpillar can be embedded into a plane satisfying the above conditions. It is one of the special cases of a conjecture by Garcia et al. [1].

1 Introduction

In this paper, we deal with finite undirected graphs with neither loops nor multiple edges. The following problem is well-known.

For given graphs $G, T_1, \cdots, T_k$, does G contain subgraphs isomorphic to $T_1, \cdots, T_k$ and pairwise edge disjoint?

This problem is NP-complete since if we suppose that G is a complete graph K_n and $k = 2$ then the problem is equivalent to SUBGRAPH ISOMORPHISM which is known to be NP-complete [2].

We say that graphs $T_1, \cdots, T_k$ can be *packed into a graph* G or *there exists a packing of* $T_1, \cdots, T_k$ *into* G if G contains subgraphs isomorphic to $T_1, \cdots, T_k$ and pairwise edge disjoint. Moreover, if it satisfies $|V(T_1)| = \cdots = |V(T_k)| = |V(G)|$, we say that $T_1, \cdots, T_k$ can be *tightly packed into* G or *there exists a tight packing of* $T_1, \cdots, T_k$ *into* G. A star with n vertices means a complete bipartite graph $K_{1,n-1}$.

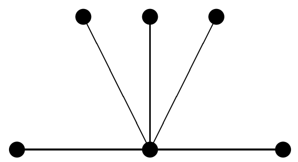

Figure 1: A star with 6 vertices ($K_{1,5}$)

[*]Department of Mathematics, Keio University, oda@math.keio.ac.jp

[†]Department of Mathematics, Keio University, ohta@math.keio.ac.jp

Hedetniemi et al. [3] proved the following theorem.

Theorem 1 (Hedetniemi et al. [3]) *Let T_1 and T_2 be non-star trees with n vertices. Then, there exists a tight packing of T_1 and T_2 into a complete graph K_n.*

Here we suppose that G is a plane graph. Garcia et al. [1] gave the following conjecture.

Conjecture 1 (Garcia et al. [1]) *Let T_1 and T_2 be non-star trees with n vertices. Then, there exists a simple plane graph G such that T_1 and T_2 can be tightly packed into G.*

If the above plane graph exists, we say that *there exists a tight planar packing of T_1 and T_2*. We observe that such a packing is equivalent to a planar drawing of T_1 and T_2 without sharing edges. In this case, we obtain a simultaneous drawing of T_1 and T_2 on the same vertex set.

The following observations are fundamental.

Observation 1 *Let T_1 be a star and T_2 be a tree with n vertices. Then, there does not exist any tight planar packing of T_1 and T_2.*

Proof. Suppose that T_1 and T_2 can be packed into a simple plane graph G. We focus on the center vertex v of the star T_1. Any edge which is incident with the vertex $\bar{v}$ of G corresponding to v must be used for T_1. Hence, we can assign no vertex of T_2 to $\bar{v}$, which is a contradiction. □

Observation 2 *There does not exist a tight planar packing of three trees.*

Proof. The sum of the number of edges for three trees with n vertices is $3(n-1)$. But the number of edges of a simple plane graph of order n is at most $3n-6$. □

Garcia et al. proved the following theorems in the same paper[1].

Theorem 2 *Let T be any tree with n vertices which is different from a star. Then, there exists a tight planar packing of T and a copy of T.*

Theorem 3 *Let T_1 be any tree with n vertices which is different from a star and let T_2 be a path of order n. Then, there exists a tight planar packing of T_1 and T_2.*

Also, we showed the following theorem.

Theorem 4 (Enomoto et al. [5]) *Let T_1 be any tree with n vertices which is different from a star. Let T_2 be a non-star graph which is obtained from a star by adding at most one vertex for each edge. (If it is obtained by adding a vertex for any edge, it is called a firework. See the lower right hand of Figure 2.) Then, there exists a tight planar packing of T_1 and T_2.*

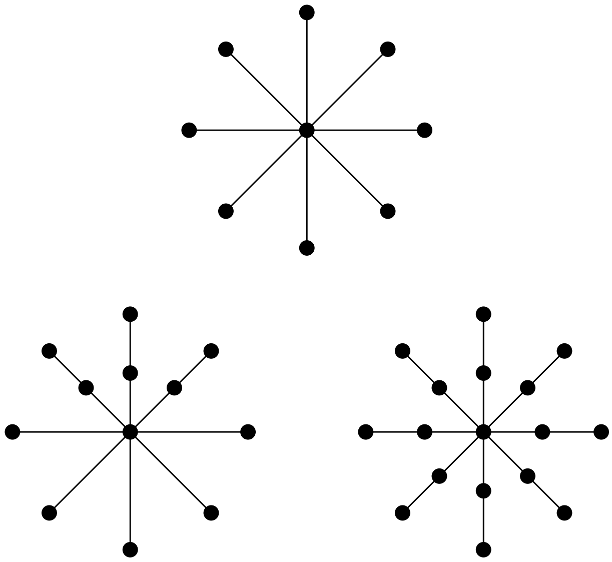

Figure 2: A star and graphs T_2 in Theorem 4

2 Main results

A caterpillar is a graph derived from a path by hanging any number of leaves from the vertices of the path.

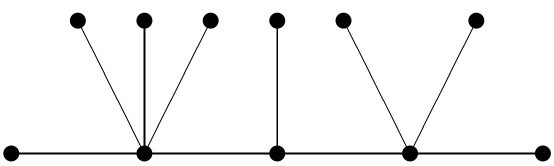

Figure 3: A caterpillar

The following is our main theorem.

Theorem 5 *Let T_1 be any tree with n vertices which is different from a star and let T_2 be any caterpillar with n vertices which is different from a star. Then, there exists a tight planar packing of T_1 and T_2.*

We give a sketch of our proof. At first, we set n vertices in some line l and each vertex is labeled from 0 to $n-1$ in linear order. Next, we consider two embeddings of T_1 and T_2 into a plane with non-crossing edges. An embedding of T_1 is to be on the upper area of l and an embedding of T_2 is to be on the lower area of l. Then, we should take care only that they do not share any edges. Here we give a key lemma as follows:

Lemma 6 *Let T_1 be a caterpillar with m vertices and T_2 be a tree with n vertices. We assume that $m < n$. Take vertices $r_i \in V(T_i)$ for $i = 1, 2$, arbitrarily. Let a be an integer such that $0 \le a \le n - m$. Then, there exist two bijection $\varphi : V(T_2) \to [0, n-1]$ and $\gamma : V(T_1) \to [a, a+m-1]$ satisfying:*

(1) Each embedding of T_1 and T_2 according to φ and γ is non-self-intersecting.

(2) The embeddings of T_1 and T_2 according to φ and γ do not share any edge.

(3) The vertex $\varphi(r_2)$ is open in $\varphi(T_2)$ and $\gamma(r_1)$ is open in $\gamma(T_1)$.

(4) $\varphi(r_2) \neq \gamma(r_1)$

(5) $\varphi(r_2) \in \{0, n-1\} \cup [a+1, a+m-2]$

We say that the vertex $\varphi(r_2)$ is *open* in $\varphi(T_2)$ if there does not exist any edge $e = (u, v)$ in T_2 such that $\varphi(u) < \varphi(r_2) < \varphi(v)$ or $\varphi(v) < \varphi(r_2) < \varphi(u)$. We define $\gamma(r_1)$ is *open* in $\gamma(T_1)$ similarly.

Theorem 5 is one of the special cases of Conjecture 1. Needless to say, our ultimate goal is to solve Conjecture 1. Toward the goal we need more general proof technique. We note that the above lemma is proved by using induction and the technique is new.

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